Wave Motion in 1-D Viscoelastic Phononic Crystals: A Comprehensive Analytical, Numerical and Experimental Study

by

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Department of Civil, Structural, and Environmental Engineering
CONTENTS

List of Figures ................................................................. v
List of Tables ................................................................. xiii
Abstract .............................................................................. xiv
Acknowledgements ........................................................... xvi

1. Introduction .................................................................... 1
   1.1 Introduction .............................................................. 1
   1.2 Background .............................................................. 3
      1.2.1 Analytical Studies ............................................... 3
      1.2.2 Numerical Studies ................................................ 4
      1.2.3 Experimental Studies .......................................... 5
         1.2.3.1 Small Amplitude Experiments ....................... 5
         1.2.3.2 Large Amplitude Excitation Based on Impact .... 7
   1.3 Motivation ................................................................. 8
   1.4 Objective ................................................................. 10
   1.5 Dissertation Outline ................................................. 10

2. Dispersion Relation of Viscoelastic Multilayered Composite in Sagittal Plane ...................................................... 13
   2.1 Introduction .............................................................. 13
   2.2 Phononic Crystal Background ....................................... 14
      2.2.1 Bravais Lattice and Reciprocal Space ................. 15
      2.2.2 Bloch Periodic Condition ................................... 16
   2.3 Oblique Wave Motion in Viscoelastic Medium ............. 17
5.4.3 Impact Excitation Tests with Hybrid SHPB .................................................. 107
5.4.4 Time-domain FE Simulations ................................................................. 109
5.5 Discussion ............................................................................................. 111
  5.5.1 Impulse-independent Wave Transmission Behavior from Harmonic Excitation Conditions .................................................. 111
  5.5.2 Impulse-dependent Wave Transmission Behavior from Impact Excitation Conditions ............................................................. 112
5.6 Conclusions ....................................................................................... 114

6. Conclusion ............................................................................................. 116
  6.1 Summary ............................................................................................. 116
  6.2 Original Contribution ........................................................................ 117
  6.3 Future Investigations ......................................................................... 118

Appendix .................................................................................................... 121

Bibliography ............................................................................................. 122
LIST OF FIGURES

2.1 (A) Infinitely periodic multilayered composite in 3-D coordinate space which is periodic along $x_3$-axis. The primitive lattice vector $a_3$ defines the unit-cell of the composite with periodic length $a_3 = ||a_3||$. (B) Unit cell of M-layered composite in 2-D sagittal plane where wave can propagate in any direction by making angle $\theta$ with the $x_3$-axis. Periodic length of the inclined wave is given by $(a_3 \cos \theta)$. (C) Reciprocal space of periodic composite showing the first Brillouin zone in $\kappa_1 - \kappa_3$ plane. The rectangle on right represents irreducible Brillouin zone (IBZ) which is bounded by $\kappa_3 \in [0, \pi/a_3]$ in $\kappa_3$-axis. The propagating wavevector $\kappa$ describes the wave motion at an angle $\theta$ which has periodic length $\pi/(a_3 \cos \theta)$ in the reciprocal space. ................................................. 14

2.2 (A) A P-wave with wavevector $\kappa_{p,1}$ (i.e. $\kappa_{p,1} = \kappa_{p,1}^R + i\kappa_{p,1}^I$) incidents at the interface between viscoelastic layers 1 and 2 with propagation angle $\theta_{p,1}$ and attenuation angle $\zeta_{p,1}$. (B-1) Reflected and transmitted P-wave denoted by subscripts 1 and 2, respectively. The propagation and attenuation vectors are represented by $\kappa_{p,j}^R$ and $\kappa_{p,j}^I$ (for $j = 1, 2$), respectively. (B-2) Illustration of reflected and transmitted SV-waves in layers 1 and 2, where propagation and attenuation wavevectors are represented by $\kappa_{s,j}^R$ and $\kappa_{s,j}^I$ (for $j = 1, 2$), respectively. The angles of propagation and attenuation waves are denoted by $\theta_{r,j}$ and $\zeta_{r,j}$ (for $j = 1, 2$ and $r = p, s$). ................................................................. 22
2.3 (A-1) Incident wavevector $\kappa_{R_{p,1}}$ at angle $\theta_{p,1}$ from the elastic medium on the interface between elastic and viscoelastic layers 1 and 2, respectively. Reflected and transmitted (A-2) P-waves and (A-3) SV-Waves which are represented by propagation wavevectors $\kappa_{R_{r,j}}$ (for $j = 1, 2$ and $r = p, s$) for both layers and the attenuation wavevectors $\kappa_{I_{r,2}}$ for the viscoelastic layer. The propagation angles in the layers are denoted by $\theta_{r,j}$ and attenuation along $x_3$-axis in the viscoelastic layer is shown by angle $\zeta_{r,2}$.

(B-1) The wavevector $\kappa_{R_{p,1}}$ incidents at the interface from a viscoelastic first layer. Reflected and transmitted (B-2) P-waves and (B-3) SV-waves with wavevectors $\kappa_{R_{r,j}}$ in viscoelastic and elastic layers, respectively. Attenuation in viscoelastic layer is represented by wavevector $\kappa_{I_{r,2}}$ with attenuation angles $\zeta_{r,1}$ in the viscoelastic medium.

2.4 Three dimensional scheme of $f - \kappa_1 - \kappa_3$ coordinate system to determine dispersion relation of viscoelastic-elastic composite. The phase dispersion relation is shown by solid line in the wave propagation plane inclined at angle $\theta$ on the $\kappa_1 - \kappa_3^R$ coordinate plane. The inclined length of the phase dispersion plane is $\pi/a_3 \cos \theta$ which has projection length of $\pi/a_3$ on the $\kappa_3^R$-axis. Three distinct points of $\kappa_0^R - f$ dispersion relation is projected on $\kappa_3^I - f$ plane which represent corresponding attenuation relation.

2.5 Viscoelastic properties of polyurethane elastomer obtained by dynamic mechanical analysis (DMA). (A) frequency dependent modulus $\hat{\lambda}(\omega)$. (B) frequency dependent modulus $\hat{\lambda}(\omega)$. The storage and loss moduli are represented by circle- and square-marked solid lines, respectively. Note that the shaded areas denote one standard deviation from two DMA tests. The constant modulus of pseudo-elastic approximation is shown by dotted lines.

2.6 Complete dispersion analysis results of pseudo-elastic composite showing attenuation relations $\kappa_3^I - f$ for wave motions at (A-1) $\theta = 0^\circ$, (B-1) $\theta = 15^\circ$, (C-1) $\theta = 30^\circ$ and (D-1) $\theta = 60^\circ$. The phase dispersion relations $\kappa_0^R - f$ are illustrated for wave motions at (A-2) $\theta = 0^\circ$, (B-2) $\theta = 15^\circ$, (C-2) $\theta = 30^\circ$ and (D-2) $\theta = 60^\circ$ where the wavevector $\kappa_0^R \in [0, \pi/(a_3 \cos \theta)]$ inside the IBZ varies with propagation angles.
2.7 Complete dispersion analysis results of viscoelastic-elastic composite showing attenuation relations $\kappa^I_3 - f$ for wave motions at (A-1) $\theta = 0^\circ$, (B-1) $\theta = 15^\circ$, (C-1) $\theta = 30^\circ$ and (D-1) $\theta = 60^\circ$. The phase dispersion relations $\kappa^R_\theta - f$ are illustrated for wave motions at (A-2) $\theta = 0^\circ$, (B-2) $\theta = 15^\circ$, (C-2) $\theta = 30^\circ$ and (D-2) $\theta = 60^\circ$ where the wavevector $\kappa^R_\theta \in [0, \pi/(a_3 \cos \theta)]$ inside the IBZ varies with propagation angles. 

2.8 Transmission coefficient $C_t$ of periodic pseudo-elastic composite calculated from (5.11) for wave motion at (A-1) $\theta = 0^\circ$, (B-1) $\theta = 15^\circ$, (C-1) $\theta = 30^\circ$ and (D-1) $\theta = 60^\circ$. Group slowness $S_g$ obtained using (2.43) for wave motion at (A-2) $\theta = 0^\circ$, (B-2) $\theta = 15^\circ$, (C-2) $\theta = 30^\circ$ and (D-2) $\theta = 60^\circ$. 

2.9 Transmission coefficient $C_t$ of periodic viscoelastic-elastic composite calculated from (5.11) for wave motion at (A-1) $\theta = 0^\circ$, (B-1) $\theta = 15^\circ$, (C-1) $\theta = 30^\circ$ and (D-1) $\theta = 60^\circ$. Group slowness $S_g$ obtained using (2.43) for wave motion at (A-2) $\theta = 0^\circ$, (B-2) $\theta = 15^\circ$, (C-2) $\theta = 30^\circ$ and (D-2) $\theta = 60^\circ$. 

3.1 (A) Unit cell of M-layered composite in 2-D sagittal plane where wave can propagate in any direction by making angle $\theta$ with the $x_3$-axis. Periodic length of the inclined wave is given by $a_3 \cos \theta$. (B-1) Reciprocal space representing wave motion perpendicular to the layer, where the top rectangle shows the first BZ. $\Gamma - X$ denotes the actual wavevector and the two aliasing wavevectors are represented by $\Gamma' - X'$ and $\Gamma'' - X''$. (B-2) Reciprocal space showing front and back aliasing zones where the inclined wavevector $\Gamma - \Lambda$ produces aliasing wavevectors $\Gamma'_F - \Lambda'_F$ and $\Gamma'_B - \Lambda'_B$, respectively. 

3.2 (A) Analytical dispersion relation for wave propagation perpendicular to 3-layered composite. Effect of aspect ratio on the numerical dispersion relation in FE framework where model aspect ratios (B-1) $a_1/a_3 = 2.0$ and (B-2) $a_1/a_3 = 1.0$ are used. 

3.3 Analytical dispersion relations of 3-layered composite for wave propagation angles (A-1) $\theta = 0^\circ$, (A-2) $\theta = 45^\circ$ and (A-3) $\theta = 75^\circ$. Effect of oblique wave motion on spectral distortion in numerical dispersion relations where same unit cell aspect ratio $a_1/a_3 = 0.4$ is used for wave propagation angles (A-1) $\theta = 0^\circ$, (A-2) $\theta = 45^\circ$ and (A-3) $\theta = 75^\circ$. 

3.4 (A) Analytical dispersion relation for wave propagation perpendicular to 2-layered composite. Effect of aspect ratio on the numerical dispersion relation in FE framework where model aspect ratios (B-1) $a_1/a_3 = 2.0$ and (B-2) $a_1/a_3 = 1.0$ are used. 

3.5 Analytical dispersion relations of 2-layered composite for wave propagation angles (A-1) $\theta = 0^\circ$, (A-2) $\theta = 45^\circ$ and (A-3) $\theta = 75^\circ$. Effect of oblique wave motion on spectral distortion in numerical dispersion relations where same unit cell aspect ratio $a_1/a_3 = 0.4$ is used for wave propagation angles (A-1) $\theta = 0^\circ$, (A-2) $\theta = 45^\circ$ and (A-3) $\theta = 75^\circ$. 

3.6 (A) Analytical dispersion relation for wave propagation perpendicular to 2-layered composite. Effect of aspect ratio on the numerical dispersion relation in FE framework where model aspect ratios (B-1) $a_1/a_3 = 2.0$ and (B-2) $a_1/a_3 = 1.0$ are used. 

3.7 Analytical dispersion relations of 2-layered composite for wave propagation angles (A-1) $\theta = 0^\circ$, (A-2) $\theta = 45^\circ$ and (A-3) $\theta = 75^\circ$. Effect of oblique wave motion on spectral distortion in numerical dispersion relations where same unit cell aspect ratio $a_1/a_3 = 0.4$ is used for wave propagation angles (A-1) $\theta = 0^\circ$, (A-2) $\theta = 45^\circ$ and (A-3) $\theta = 75^\circ$. 

3.8 (A) Analytical dispersion relation for wave propagation perpendicular to 2-layered composite. Effect of aspect ratio on the numerical dispersion relation in FE framework where model aspect ratios (B-1) $a_1/a_3 = 2.0$ and (B-2) $a_1/a_3 = 1.0$ are used. 

3.9 Analytical dispersion relations of 2-layered composite for wave propagation angles (A-1) $\theta = 0^\circ$, (A-2) $\theta = 45^\circ$ and (A-3) $\theta = 75^\circ$. Effect of oblique wave motion on spectral distortion in numerical dispersion relations where same unit cell aspect ratio $a_1/a_3 = 0.4$ is used for wave propagation angles (A-1) $\theta = 0^\circ$, (A-2) $\theta = 45^\circ$ and (A-3) $\theta = 75^\circ$. 

3.10 (A) Analytical dispersion relation for wave propagation perpendicular to 2-layered composite. Effect of aspect ratio on the numerical dispersion relation in FE framework where model aspect ratios (B-1) $a_1/a_3 = 2.0$ and (B-2) $a_1/a_3 = 1.0$ are used. 

3.11 Analytical dispersion relations of 2-layered composite for wave propagation angles (A-1) $\theta = 0^\circ$, (A-2) $\theta = 45^\circ$ and (A-3) $\theta = 75^\circ$. Effect of oblique wave motion on spectral distortion in numerical dispersion relations where same unit cell aspect ratio $a_1/a_3 = 0.4$ is used for wave propagation angles (A-1) $\theta = 0^\circ$, (A-2) $\theta = 45^\circ$ and (A-3) $\theta = 75^\circ$.
3.4 (A-1) Complete wavevector \( \Gamma - \Lambda \) inclined at \( \theta = 75^\circ \) in reciprocal space which has \( \kappa_1 \) component larger that BZ limit \( \pi/a_1 \). (A-2) Wavevector segment \( \Gamma - Y \) within first BZ which is intersected by BZ boundary. (A-3) Remaining wavevector segment \( Y' - \Gamma' \) which is shifted to the first BZ. (B-1) Dispersion relation of complete wavevector \( \Gamma - \Lambda \) which is obtained by combining the dispersion relations of (B-2) wavevector segment \( \Gamma - Y \) and (B-2) wavevector segment \( Y' - \Gamma' \). . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 48

3.5 (A) [Left] A unit cell of the infinitely periodic 3-layered composite having \( a_1/a_3 = 2.0 \), which is employed for numerical dispersion analysis. [Right] The corresponding wavevector domain, which illustrates the valid wavevector path for waves perpendicular to the layers (\( \Gamma - X \)) and the corresponding aliasing paths (\( \Gamma' - X', \Gamma'' - X'' \)). (B) FE dispersion relation obtained by employing a unit cell of \( a_1/a_3 = 2.0 \). Note that the numerical dispersion relation contains unwanted fictitious modes represented by red solid lines. (C) Three analytical dispersion relations obtained from (2.42): (C-1) \( \kappa_1 = 0 \), (C-2) \( \kappa_1 = 2\pi/a_1 \), and (C-3) \( \kappa_1 = 4\pi/a_1 \). (D) The projection of all the analytical dispersion relations in Fig. 3.5C onto the \( \kappa_3 - \omega \) plane. Note that one can find one-to-one map in all the observed wave modes in Figs. 3.5B and 3.5D, indicating that the fictitious modes originate from the aliasing paths. Moreover, the maximum valid frequency \( \omega_{\text{max}} \) from (3.16) is denoted by a line with square markers. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 54

3.6 (A) [Left] A unit cell of the infinitely periodic 3-layered composite having \( a_1/a_3 = 1.0 \), which is employed for numerical dispersion analysis. [Right] The corresponding wavevector domain, which illustrates the valid wavevector path for waves perpendicular to the layers (\( \Gamma - X \)) and the corresponding aliasing paths (\( \Gamma' - X', \Gamma'' - X'' \)). (B) FE dispersion relation obtained by employing a unit cell of \( a_1/a_3 = 1.0 \). Note that the numerical dispersion relation contains unwanted fictitious modes represented by red solid lines. (C) Three analytical dispersion relations obtained from (2.42): (C-1) \( \kappa_1 = 0 \), (C-2) \( \kappa_1 = 2\pi/a_1 \), and (C-3) \( \kappa_1 = 4\pi/a_1 \). (D) The projection of all the analytical dispersion relations in Fig. 3.6C onto the \( \kappa_3 - \omega \) plane. Note that one can find one-to-one map in all the observed wave modes in Figs. 3.6B and 3.6D, indicating that the fictitious modes originate from the aliasing paths. Moreover, the maximum valid frequency \( \omega_{\text{max}} \) from (3.16) is denoted by a line with square markers. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 55
3.7  (A) [Left] A unit cell of the infinitely periodic 4-layered composite having $a_1/a_3 = 2.0$, which is employed for numerical dispersion analysis. [Right] The corresponding wavevector domain, which illustrates the valid wavevector path for waves perpendicular to the layers ($\Gamma - X$) and the corresponding aliasing paths ($\Gamma' - X'$, $\Gamma'' - X''$). (B) FE dispersion relation obtained by employing a unit cell of $a_1/a_3 = 2.0$. Note that the numerical dispersion relation contains unwanted fictitious modes represented by red solid lines. (C) Three analytical dispersion relations obtained from (2.42): (C-1) $\kappa_1 = 0$, (C-2) $\kappa_1 = 2\pi/a_1$, and (C-3) $\kappa_1 = 4\pi/a_1$. (D) The projection of all the analytical dispersion relations in Fig. 3.7C onto the $\kappa_3 - \omega$ plane. Note that one can find one-to-one map in all the observed wave modes in Figs. 3.7B and 3.7D, indicating that the fictitious modes originate from the aliasing paths. Moreover, the maximum valid frequency $\omega_{\text{max}}$ from (3.16) is denoted by a line with square markers. ................................................................. 56

3.8  (A) [Left] A unit cell of the infinitely periodic 4-layered composite having $a_1/a_3 = 1.0$, which is employed for numerical dispersion analysis. [Right] The corresponding wavevector domain, which illustrates the valid wavevector path for waves perpendicular to the layers ($\Gamma - X$) and the corresponding aliasing paths ($\Gamma' - X'$, $\Gamma'' - X''$). (B) FE dispersion relation obtained by employing a unit cell of $a_1/a_3 = 1.0$. Note that the numerical dispersion relation contains unwanted fictitious modes represented by red solid lines. (C) Three analytical dispersion relation obtained from (2.42): (C-1) $\kappa_1 = 0$, (C-2) $\kappa_1 = 2\pi/a_1$, and (C-3) $\kappa_1 = 4\pi/a_1$. (D) The projection of all the analytical dispersion relations in Fig. 3.8C onto the $\kappa_3 - \omega$ plane. Note that one can find one-to-one map in all the observed modes in Figs. 3.8B and 3.8D, indicating that the fictitious modes originate from the aliasing paths. Moreover, the maximum valid frequency $\omega_{\text{max}}$ from (3.16) is denoted by a line with square markers. ................................................................. 57
3.9  (A) [Left] A unit cell of the infinitely periodic 3-layered composite having $a_1/a_3 = 0.4$ where wave propagates at $\theta = 45^\circ$. [Right] The corresponding wavevector domain, which illustrates the valid wavevector path $\Gamma - \Lambda$ and the corresponding front and back aliasing paths $\Gamma'_F - \Lambda'_F$, $\Gamma'_B - \Lambda'_B$, respectively. (B) FE dispersion relations where the numerical dispersion relation contains unwanted fictitious modes represented by red solid lines. (C) Three analytical dispersion relation obtained from (2.42) for wavevectors (C-1) $\Gamma - \Lambda$, (C-2) $\Gamma'_B - \Lambda'_B$, and (C-3) $\Gamma'_F - \Lambda'_F$. (D) The projection of all the analytical dispersion relations in Fig. 3.9C onto the $\kappa_\theta - \omega$ plane. Note that one can find one-to-one map in all the observed modes in Figs. 3.9B and 3.9D, indicating that the fictitious modes originate from the aliasing paths. Moreover, the maximum valid frequency $\omega_{\text{max}}$ from (3.16) is denoted by a line with square markers. ..................................................... 60

3.10  (A) [Left] A unit cell of the infinitely periodic 3-layered composite having $a_1/a_3 = 0.4$ where wave propagates at $\theta = 75^\circ$. [Right] The corresponding wavevector domain, which illustrates the valid wavevector path $\Gamma - \Lambda$ and the corresponding front and back aliasing paths $\Gamma'_F - \Lambda'_F$, $\Gamma'_B - \Lambda'_B$, respectively. (B) FE dispersion relations where the numerical dispersion relation contains unwanted fictitious modes represented by red solid lines. (C) Three analytical dispersion relation obtained from (2.42) for wavevectors (C-1) $\Gamma - \Lambda$, (C-2) $\Gamma'_B - \Lambda'_B$, and (C-3) $\Gamma'_F - \Lambda'_F$. (D) The projection of all the analytical dispersion relations in Fig. 3.10C onto the $\kappa_\theta - \omega$ plane. Note that one can find one-to-one map in all the observed modes in Figs. 3.10B and 3.10D, indicating that the fictitious modes originate from the aliasing paths. Moreover, the maximum valid frequency $\omega_{\text{max}}$ from (3.16) is denoted by a line with square markers. ..................................................... 61

3.11  (A) Unit cell of M-layered composite in 2-D sagittal plane where $a_1$ and $a_3$ denotes the lattice vectors. The periodic length of the inclined wave is given by $a_3 \cos \theta$. (B) The front and back aliasing wavevectors $\Gamma'_F - \Lambda'_F$ and $\Gamma'_B - \Lambda'_B$ in reciprocal space which located at $2\pi/a_1$ from corresponding points of actual wavevector $\Gamma - \Lambda$ along $\kappa_1$-axis. The smallest aliasing wavenumber is represented by the path $X - \Lambda'_F$ at distance $2\pi/a_1 - \kappa'_1$ from $\kappa_3$-axis. ..................................................... 63

4.1  Schematic of basic SHPB set-up showing input bar, output bar, striker and specimen. ..................................................... 70
4.2 Comparison of stress-strain relations of polymers at low strain rate .... 71
4.3 Comparison of stress-Strain relations of polymers at high strain rate .... 72
4.4 Complete aluminum and nylon SHPB in SEESL laboratory. ............... 72
4.5 Nylon striker velocity at different pressure conditions in launcher which are obtained analytically for (A) varying acceleration (B) constant acceleration ... 75
4.6 Schematic of gas supply system for striker launcher ...................... 76
4.7 Valve Outlet and Striker Guide Inlet ..................................... 79
4.8 Laser based photoelectric sensor for measuring striker velocity .......... 81
4.9 (A) Top view and (B) front view of tapered nylon striker ................. 83
4.10 (A) Strain signals from radially opposite sides of a SHPB bar (B) averaged signal at the same location representing only the strain due to longitudinal wave ... 84
4.11 (A) Incident and reflected strain readings of single bar experiment by a 250 mm striker and the corresponding (B) dispersion relation between phase velocity and frequency, and (C) frequency dependent attenuation coefficient ............. 86
4.12 (A) Phase velocity and (B) attenuation coefficient representing the propagation coefficient of aluminum bar. Propagation coefficient of nylon bar showing (C) phase velocity and (D) attenuation coefficient. The average experimental results are denoted by solid lines and standard deviations are represented by shaded areas.
In addition, analytical solution is shown by dashed line in phase velocity spectra. 88
4.13 Transferred input and output strain signals at the contact interface generated by (A) 150 mm striker with impact velocity 4.47m/s, (B) 200 mm striker with impact velocity 4.71m/s , (C) 250 mm striker with impact velocity 4.75m/s and (D) 300 mm striker with impact velocity 4.48m/s. ......................... 90

5.1 Viscoelastic properties of the considered silicone rubber. (A) Loading-unloading stress-strain relation, whose equilibrium path is captured by the Yeoh model. (B) Frequency dependent storage modulus \( \hat{E}'(\omega) \) and loss modulus \( \hat{E}''(\omega) \). (C) Time dependent relaxation modulus. .......................... 95
5.2 (A) Geometry of the considered phononic crystal specimen. (B) The corresponding axisymmetric FE model, whose symmetric center-line is denoted by the dotted line. 96
5.3 Pressure wave characteristics of the infinitely periodic viscoelastic-elastic phononic crystal under consideration. (B) Phase dispersion relation, \( \kappa_{pc}^R - f \). (B) Attenuation relation, \( \kappa_{pc}^I - f \). (C) Transmission coefficient obtained from (5.11), \( C_t \). ............. 100
5.4 (A) Electrodynamic shaker test set-up. (B) SHPB test set-up. .......................... 102

5.5 Amplitude-independent transmission characteristics. (A) Transmission coefficient $C_t(\omega)$ obtained from the base excitation tests with electrodynamic shaker: (A-1) 21.0 N, (A-2) 53.7 N. Note that the shaded area denotes the standard deviation. (B) The corresponding results from the time-domain FE simulations: (B-1) 21.0 N, (B-2) 53.7 N. ................................................................. 103

5.6 Wave propagation characteristics of SHPB apparatus bars. (A) Aluminum bar: (A-1) phase velocity $c_{al}(\omega)$, (A-2) attenuation coefficient $\kappa_{al}(\omega)$. (B) Nylon bar: (B-1) phase velocity $c_{ny}(\omega)$, (B-2) attenuation coefficient $\kappa_{ny}(\omega)$. Note that the average experimental results are denoted by the solid lines and the standard deviations are represented by the shaded areas. In addition, the dashed and the dotted lines indicate Pochhammer-Chree analytical solution and the numerically-obtained propagation coefficient from FE simulations, respectively. ............................... 107

5.7 Spectrum of input strain signals experimentally obtained from different striker lengths. ................................................................. 108

5.8 Amplitude-dependent transmission characteristics obtained from SHPB setting. (A) Transmission coefficient $C_t(\omega)$ from the SHPB tests: (A-1) 50 mm-long striker with the impulse of 5.6 $N \cdot s$, (A-2) 150 mm-long striker with the impulse of 13.2 $N \cdot s$, and (A-3) 250 mm-long striker with the impulse of 18.6 $N \cdot s$. (B) The corresponding results from the time-domain FE simulations: (B-1) 50 mm-long striker with the impulse of 5.5 $N \cdot s$, (B-2) 150 mm-long striker with the impulse of 13.6 $N \cdot s$, and (B-3) 250 mm-long striker with the impulse of 18.5 $N \cdot s$. (C) The numerical results obtained from the additional time-domain FE simulations where a 150mm-long striker is launched for all three cases: (C-1) the impulse of 5.5 $N \cdot s$, (C-2) the impulse of 13.6 $N \cdot s$, and (C-3) the impulse of 18.7 $N \cdot s$. .... 110

5.9 Contour plots of transmission coefficient with respect to input impulse and frequency, showing the evolution of impulse-dependent wave transmission characteristics. (A) $C_t(\omega) = \|A_{out}(\omega)\|/\|F_{in}(\omega)\|$ defined in (5.11). (B) $\tilde{C}_t(\omega) = \|F_{out}(\omega)\|/\|F_{in}(\omega)\|$. Note that the dark brown color indicates the low transmission frequency zones. ................................................................. 114
LIST OF TABLES

3.1 Properties of considered materials ............................................. 52

4.1 Impedance Comparison of Polymeric Materials ............................. 71

4.2 Properties of nitrogen at 185 psi from software ........................... 78

5.1 Prony series coefficients of the considered silicone rubber .......... 94

5.2 Specifications of the hybrid SHPB apparatus. ............................... 106
Phononic crystals are engineered materials which exhibit unique wave motion properties due to periodic arrangement of multiple homogeneous constituents. Over the last decade, metallic phononic crystals have been thoroughly investigated. However, viscoelastic phononic crystals composed of polymers and metals have not been thoroughly investigated although they can be adopted for potential applications in the acoustic frequency range (i.e., $< 20\text{kHz}$). For instance, the analytical dispersion relation of oblique waves has not been reported in the sagittal plane of 1-D viscoelastic phononic crystals. There is also a long standing issue of fictitious modes in the numerical dispersion relation of 1-D phononic crystals in the finite element (FE) framework. Furthermore, despite the rising interest in the nonlinear wave transmission characteristics of viscoelastic phononic crystals, conventional experimental test setups for phononic crystal studies (e.g., electrodynamic shakers and piezoelectric actuators) are not suitable to generate sufficiently large excitation to induce nonlinear wave motion.

This study presents comprehensive analytical, numerical and experimental investigations to explore the fundamental wave characteristics of 1-D viscoelastic phononic crystals. Firstly, the closed-form dispersion relations of 1-D viscoelastic phononic crystals is derived for oblique wave motion in the sagittal plane. Secondly, the issue of fictitious modes in the numerical dispersion relations in the FE framework is solved by investigating the spatial aliasing originated from the artificial periodicity in FE models. Furthermore, spectral distortions in the numerical dispersion relations can be avoided by considering a generalized guideline based on an anti-aliasing condition and an effective modulus theory. Lastly, this study propose a hybrid split Hopkinson pressure bar (SHPB) apparatus as a tool to identify the nonlinear wave characteristics of 1-D viscoelastic phononic crystals. For the considered 1-D viscoelasatic phononic crystals, the application of the hybrid SHPB apparatus reveals new low transmission frequency zones, which are neither predicted from the analytical solution nor observed from the conventional electrodynamic shaker
tests. This experimental study demonstrates the impulse-dependent wave transmission behavior can be investigated by adopting a hybrid SHPB apparatus.

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1. INTRODUCTION

1.1 Introduction

Phononic crystals are engineered metamaterials which have attracted the researchers due to the prospect of controlling their wave propagation properties. The mechanical behavior of these materials are obtained by macroscopic periodic arrangement of conventional metals, polymers or plastics in different configurations. The noble wave propagation properties of phononic crystals originate from acoustic impedance mismatch of the constituent materials. Their precise shape, geometry, size and orientation can affect the acoustic and elastic waves in an unconventional manner, creating material properties which are unachievable with conventional materials. The most notable feature of phononic crystals is the frequency band-gap in dispersion relations which refers to attenuation of elastodynamic wave at certain frequency ranges. Since crystal behavior depends not only on the properties of the constituent materials, but also the pattern at which the constituent materials are arranged, the phononic band-gaps can be manipulated by controlling the lattice size, orientation and constituent materials. As a result, investigations on wave motion in phononic crystals with various configurations have gained popularity in recent years.

The wave propagation in the of 1-D phononic crystals was first investigated \cite{152, 157} to characterize their effective modulus properties. Moreover, these crystals which are also referred as periodic layered composites, are most commonly used phononic crystals for various application purposes. These applications include phononic band-gaps \cite{6, 40, 84, 85, 92, 150, 205}, negative effective dynamic properties \cite{133, 207}, stress-wave attenuation \cite{134}, controlling thermal conductivity \cite{94, 146}, nano-structured polymer layers \cite{46}, acoustic rectifier \cite{100, 104}, acoustic waveguide \cite{158}, etc. Despite significant investigations on layered composites, several aspects of this field requires further attentions.

The analytical dispersion relation for plane wave propagation in sagittal plane in viscoelastic infinitely periodic layered composites (IPMC) is not completely developed. The dispersion
relation of viscoelstic composite is only studied for wave propagation perpendicular to the layers \([126, 183]\). On the other hand, due to the complexity of the formulation, the analytical investigations on dispersion relation for wave motion at arbitrary angles in the sagittal plane is not performed. The intricacy in the analysis originate from coupling of pressure wave (P-wave) and vertical shear wave (SV-wave) in sagittal plane. Note that the previous studies \([126, 183]\) has illustrated significant deviation in dispersion relation of viscoelasticity composites compared to their elastic counterpart even for perpendicular wave motion. Therefore, the dispersion relation of viscoelastic IPMC should be investigated to evaluate the viscoelastic effects on dispersion relations.

On the other hand, the several numerical dispersion analyses have endured the long standing issue of fictitious modes \([2, 70, 116, 117, 127]\). The appearance of these modes in numerical dispersion relation are similar to the actual modes which renders the spectral distortion indistinguishable. Although, researchers have noticed the issue \([116, 117]\), the source of the fictitious modes remains unresolved. Since, The source of this problem remains unresolved, the common approach is to heuristically identify the fictitious modes by exploring different unit cell modes. However, such method of analysis has the potential for leading to erroneous outcome. Therefore, the proper attention should be paid to identify the source of spectral distortion of numerical dispersion relations.

Even though the periodic crystals provide the prospect of controlling wave transmission behavior according to requirement, the primary application of such crystals are limited within small amplitude excitations. For assessment of the actual behavior of phononic crystals, vibration shaker and transducer generated harmonic loads are commonly employed experimental methods in the research community. Note that these equipments can only generate small amplitude waves which render them inapplicable for large excitation tests. Specifically, the phononic crystals which consists of soft viscoelastic rubber can undergo finite deformation due to large impact load. Importantly, the wave propagation behavior in these crystals can reach nonlinear regime which cannot be analyzed by existing analytical and numerical methods. Alternatively, experimental determination of wave transmission property can be regarded as the feasible option. But, the common experimental systems are incapable of applying necessary large amplitude wave which inhibit the progress of this field.
1.2 Background

This section briefly describes the current state of analytical, numerical and experimental studies pursued in the field of infinitely periodic layered composites. Based on the progress in the respective area, the concerning issues which require proper attention are explained in the following subsections.

1.2.1 Analytical Studies

Due to the broad prospect of periodic layered composites, different aspects of analytical dispersion relations of one dimensional periodic systems (i.e., layered composites) has been studied substantially. For instance, the analytical dispersion relation for wave propagation perpendicular to the layers is investigated thoroughly for elastic infinitely periodic bilayered composites (IPBC) \[3, 31, 77, 93, 157, 180\] and IPMC \[58, 84, 182\]. Several investigations \[137, 138, 155, 160\] are conducted on the analytical dispersion relations of sagittal plane wave in IPBC. However, very limited studies \[30, 130\] have focused on dispersion relation of oblique wave propagation in the sagittal plane of IPMC. Furthermore, the formulation procedure followed in those studies are unduly complicated. Therefore, a generalized formulation for analytical dispersion relation of sagittal plane wave in multilayered composite is necessary which is viable for practical uses.

Note that the elastic layered composites composed of metals show unique wave characteristics typically at high frequency ranges (e.g., MHz or higher). However, most of the vibrations and noises concerning human environment are characterized by low frequency ranges (e.g., a few kHz or lower) \[9, 144, 187\], especially in sonic frequency range \(< 20kHz\). In order to implement the phononic wave motion properties of periodic crystals to practical dynamic applications, researchers commonly choose polymeric materials as a constituent of composites and exploit the high impedance mismatch between metals and polymers. Notably, the polymeric materials used in the periodic composites are viscoelastic in nature. Based on temperature and frequency dependent response, these polymers are characterized as thermoplastic and thermoset. The flexible thermosets are most commonly used with metals to obtain high impedance mismatch. However, the thermoset rubber with extremely low stiffness can have sufficiently high damping at sonic frequency range \[55, 112, 143, 156\]. Since the modulus of these polymers are frequency dependent, wave propagation through the periodic composites can be significantly affected by the inherent viscosity of polymer.
By recognizing the influence of damping attenuation, several researchers have considered the viscoelasticity of polymers to study the dispersion relation of layered composites for wave propagation perpendicular to the layers. For instance, using frequency-dependent complex moduli, Tanaka and Kon-No [183] analytically studied the dispersion relations of waves perpendicular to the layers in infinitely periodic layered composites composed of viscoelastic and elastic solids (i.e., viscoelastic-elastic layered composites) [183]. In addition, several numerical methods are also employed to determine the complex dispersion relation for wave propagation perpendicular to viscoelastic composite using finite difference method [121], variational method [122], Fourier expansion method [47], finite element method [127] and PWE method [206], etc. Noticeably, no investigation is concentrated on analytical dispersion relation for wave propagation at arbitrary direction in the sagittal plane of periodic viscoelastic layered composites. Note that a group of researchers have intensively studied the transmission and reflection of oblique wave motion in viscoelastic medium by addressing the intricate wave attenuation characteristics at the viscoelastic layer interfaces [24–27, 51, 89, 161, 165, 167, 175]. However, the effect of attenuation angle is investigated only for finite multilayered composites where infinite periodicity is ignored, so the complete dispersion relation cannot be obtained from these studies. Nonetheless, outcome of their research has shown that similar to elastic composite, the P-wave and SV-wave attains mixed polarization in sagittal plane and the dispersion relation can only be obtained using coupled mode analysis. Further complexity is added by damping of viscoelastic medium which exhibit distinct wave motion behavior at viscoelastic interface [25, 27, 50, 102]. Consequently, the formulation of oblique wave propagation becomes rather intricate compared to elastic counterpart.

1.2.2 Numerical Studies

In order to analyze various practical applications, several numerical approaches have been introduced to investigate wave motion of various types of periodic composites in parallel with analytical studies. Numerical techniques for obtaining phononic dispersion relations include the continuum power series method [79, 80], the effective stiffness method [82, 178, 179], the mixture theory [123–125], the plane wave expansion method [206], the finite difference method [122], the variational method [88, 132] and the conventional FE method [2, 70, 190]. In the midst of these numerical approaches, FE method stands out for several reasons. First, the aforementioned numerical studies are performed using bilayered composites and adapting some of those methods to multilayered composite can be rather intricate. This is not the issue for
FE-method, which can be easily extended to perform dispersion analysis of periodic multilayered composites. Second, only few of the investigations considered wave propagation oblique to the layers such as, the studies for effective stiffness [178], mixture theory [123, 125, 178], continuum power series method [79] and the variational method [116]. As a matter of fact, FE method can be also used to determine dispersion relation for wave propagation oblique to the layers, which will be explained in this study. Third, the nonlinear geometry and material aspects of a system can be easily modeled in the FE framework. Therefore, FE method has been adopted to efficiently obtain phononic dispersion relations for a wide range of geometries and materials: dispersion relations of homogeneous medium [10, 18, 70, 72, 91, 131, 186], dispersion relations of layered composites [2, 70, 117, 127], 2-D periodic medium’s dispersion relations in the out-of-plane direction [18, 19, 96], 2-D periodic medium’s dispersion relations in the in-plane direction [15, 16, 101, 120, 171, 176, 190, 195, 204], 3-D periodic medium’s dispersion relations [11, 95, 97, 98, 189], etc. In particular, the FE method offers a remarkable framework to efficiently model material and geometric nonlinearity to investigate band-gap tuning of phononic dispersion relations [15, 120, 171, 189, 190].

1.2.3 Experimental Studies

Several studies are conducted to experimentally characterize wave transmission behavior of phononic crystals. These experiments are commonly performed by employing vibration shaker or transducer to apply small amplitude excitation on one-dimensional (1-D), two-dimensional (2-D), and three-dimensional (3-D) crystal settings. However, limited amount of investigation is conducted to characterize impact based wave motion property of phononic crystal which is also discussed in this section.

1.2.3.1 Small Amplitude Experiments

Experiments Using Vibration Shaker

Vibration shakers provide a reliable excitation source at several kHz level frequencies which are efficient for studying phononic crystals with large unit cells. For instance, in the formation of 1-D crystal, Wen et al. [192] determined band-gap of aluminum and copper based mass-spring system up to 2 kHz frequency. Another 1-D phononic crystal comprised of copper and rubber was investigated by Yan-Lin et al. [197] to obtain band-gap for longitudinal excitation. However,
the tests between $0 - 2 \, kHz$ frequency was unable to identify any band-gap due to high damping of the rubber. The effect of fluid structure interaction on the band-gap property of 1-D periodic airfoil was investigated by Casadei and Bertoldi [41]. The study up to 0.05 kHz showed that the band-structure property of the periodic airfoil was highly dependent on the airflow velocity. An 1-D waveguide was experimented to determine the attenuation property of flexural wave by Domadiya et al. [56] using two types of periodic beam system. The elastic steel and aluminum beams were excited by a suspending shaker to apply linear sweep between $0.1 - 6 \, kHz$ and several flexural band-gaps were identified. For a composite 1-D pipe made of periodic repetition of steel and polymer, Wen et al. [191] performed vibration experiment using a white noise signal between bandwidth $0.0 - 3.2 \, kHz$. This flexural wave experiment also provided expected band-gap property of the pipe system. On the other hand, a 2-D phononic crystal was studied by Shan et al. [163] to obtain tunable elastic band-gap. Vibration up to 3.0 kHz was applied using an electrodynamic shaker on various deformed shape of the specimen evaluate band-gap tuning. Another phononic band-gap study was performed on 2-D periodic crystals where steel and rubber discs were periodically arranged as a scatters on aluminum matrix. Again, vibration exciter was utilized to excite the system in low frequency range up to 1 kHz to find the transmission spectra.

**Experiments Using Vibration Piezoelectric Transducer**

Piezoelectric transducers are able to generate high frequency vibrations, even at few MHz level. As a result, transducers are best suited for testing phononic crystals with small unit cell which are characterized by useful wave motion behavior at high frequency range.

One of the early experiments of 1-D periodic crystal was performed by Binson and Leppelmeier [154] on steel-copper composite. Phononic band-gap of the crystal was evaluated up to 15 MHz by transducer using pulse-echo and long-pulse methods. Another noteworthy experiment on acoustic band-gap of 1-D multilayered composite was conducted by Hayashi et al. [76]. Several composites made from various combinations of steel, aluminum, copper and PVC was experimented to determine periodicity induced wave attenuation. Notably, excitation of frequency up to 40 MHz was applied and measured using piezoelectric transducers. In addition, the omnidirectional attenuation of periodic layered composite was evaluated by Martinez et al. [108]. A composite consisting of lead and epoxy layers was found to possess complete omnidirectional band-gap from the experiments conducted for frequency range 600 kHz. Beside layered composites, periodic discrete chain systems were also studied for phononic behavior. Meidani et al. [110] used
cylindrical quartz particles in different periodic arrangements to form phononic crystals. An actuator was used for generating white noise with bandwidth \(0 - 14 \text{kHz}\) to find frequency band-gaps of various crystal formations.

Due to simple experimental set-up, piezoelectric transducers are also considered for the study of 2-D and 3-D phononic crystals. Yu et al. [202] performed experiments on 2-D locally resonant phononic crystals composed of thin plates containing lead-epoxy composite. Transmission coefficients of these crystals was determined for the range \(0 - 3 \text{kHz}\). In another study, a 2-D phononic crystal was investigated by Pachiu [142] to find the nonlinear dispersion effect. The phononic crystal used for experiment was made from a nylon block with cylindrical hole in square pattern. A set of piezoelectric transducers (1 MHz central frequency) was used to generate and receive signals on opposite faces of the crystal which illustrated the second harmonic generation. By employing laser vibrometry technique, the wavefield of a 2-D phononic crystal was reconstructed by Celli and Gonella [42]. A periodic hexagonal lattice structure made of aluminum was excited by transducer to produce the necessary vibration for the operation. More recently, attention has been paid to the experimental investigation on 3-D phonic crystal. Merkel et al. [113] used steel spheres in a hexagonal closed pack (hcp) arrangement for creating 3-D. The wave motion up to 10 kHz was inserted in the crystal a transducer and the response was captured using vibrometer to obtain the attenuation spectra. In another endeavor, the rotational elastic wave in a 3-D granular crystal was demonstrated by Merkel et al. [114] to prove Cosserat’s theory. In this particular experiment, the operation of both transmitter and receiver was carried out with similar transducer for frequency bandwidth \(0 - 200 \text{kHz}\).

### 1.2.3.2 Large Amplitude Excitation Based on Impact

From the present discussion, it is evident that the experimental studies of phononic crystals using vibration shaker and transducer are suitable for small scale harmonic excitation. However, many practical applications are capable of employing short duration impact loads. In contrast to the harmonic loading conditions, impact induced wave propagation can instigate wide variety of behaviors. Due to intricate wave motion properties under impact loading, a proper experimental guideline has not developed which impede the potential applications. Nonetheless, few impact based tests are performed on phononic crystals which are presented here.

Majority of the impact based experiments are performed on discrete granular system. One such investigation was conducted by Herbold et al. [81] to characterize various types of nonlinear waves
such as, quasiharmonic, solitary and shock waves in 1-D chain. Different levels of nonlinearity (e.g., linear, weakly nonlinear and strongly nonlinear) were generated by $Al_3O_3$ cylindrical striker in diatomic periodic chain composed of PTFE sphere and steel cylinder. The experimental results showed impact induced wave propagation inside the conventional frequency band-gap zone. Moreover, few other studies were conducted on similar experimental settings where different types of waves were created in discrete chains [53, 136]. Another experiment on 1-D discrete periodic chains was conducted by Yang and Dario [198] to find the amplitude dependent band-gap property of discrete periodic chains. The evolution of linear band-gap to nonlinear solitary wave in the crystal was studied by varying the geometric configuration and impact conditions. Beside different impact forces from aluminum striker, precompression on the crystal was also used as a source of nonlinearity. This study eventually found that the linear band-gaps disappear at high impact load. In the 2-D crystal settings, nonlinear wave propagation in granular crystal was investigated by Yang and Sutton [199]. This particular experimental research illustrated the importance of rotational dynamics which was numerically captured by employing Hertz-Mindlin contact between granular particles. On the other hand, impact based experiment on continuum system is further scarce. A notable set of experiments were performed on periodic layered composite by Feng and Liu [60, 61]. These studies utilized steel split Hopkinson pressure bar (SHPB) system to evaluate the effect of initial stress and confining pressure. The tests were conducted on steel-epoxy and aluminum-epoxy composites by varying the initial parameters. However, the study did not consider the effect on impact amplitude, as all the outcomes were presented in linear wave propagation platform.

1.3 Motivation

Based on the discussion presented in Sec. 1.2, it is apparent that the 1-D phononic crystal require adequate attention in all the analytical, numerical and experimental discipline.

The elastic composites are primarily composed of high impedance metals which causes the valuable frequency dispersion properties to rise in the ultrasonic range. In order to overcome the drawback, low stiffness viscoelastic materials are essential. However, the viscoelastic damping of rubber can significantly alter the dispersion relation which is illustrated for wave propagation perpendicular to the layers. In addition, several numerical studies conducted on such composite shows viscoelasticity induced deviation in dispersion relation for wave motion perpendicular to
the layers. Notably, the effect of viscous attenuation has never been studied for oblique wave propagation in sagittal plane of viscoelastic IPMC. A investigation is required to determine the comprehensive dispersion relation which will illustrate actual behavior of viscoelastic IPMC.

FE method is most convenient numerical approach for incorporating material and geometric nonlinearities. However, in the FE framework, the accuracy of numerical analysis is highly impaired by the existence of fictitious modes [2, 116, 117]. Although, it is observed that the appearance of the fictitious modes in dispersion relation is indeed related to the aspect ratio of the unit cell model. Especially, the fictitious modes rises to higher frequency ranges with reduction of model height. In order to avoid the spectral distortion, arbitrary thin unit cells are commonly considered during analysis [2, 70, 117, 127]. Notably, the source of the fictitious mode is still unknown to the researchers. In addition, the dispersion relation of periodic composite for oblique wave motion is not investigated in the FE framework and the behavior of corresponding spectral distortion is also unexplored. Due to intricate nature of oblique wave motion, the appearance of fictitious modes can be much more complicated. Therefore, the source of fictitious modes in numerical dispersion relation should be identified. In addition, an appropriate guideline is required to obtain accurate dispersion relation.

Currently, there is a lack of experimental systems to evaluate the amplitude-dependency of layered composites. Low stiffness viscoelastic polymers are necessary in the crystal for most applications which are characterized by low frequency ranges. In practical field, these soft rubber materials can undergo large deformation and thus require impact based experimental systems. The experimental studies of phononic crystals explained in Sec. 1.2.3 are primarily performed using electromagnetic shakers [41, 56, 78, 149, 163, 191, 197] and transducers [42, 76, 108, 110, 113, 142, 154, 202]. Generally, within the frequency zone of interest, these equipment can apply displacements in the regime of micrometer or even less. Nevertheless, few studies [22, 38, 53, 81, 107, 136, 151, 198, 199] are conducted experimentally to assess the amplitude-dependent wave propagation in phononic crystals, which are predominantly based on discrete structure (i.e., granular crystals or mass spring system). Evidently, experimental investigation of large amplitude wave in continuum phononic crystals has not been performed. A systematic procedure to evaluate the amplitude-dependent wave motion property of phononic crystal is necessary to benefit wide ranging applications.
1.4 Objective

The primary aim of this dissertation is to characterize the wave propagation property of periodic layered composites. Since the study approaches analytical, numerical and experimental issues, the objectives can be subdivided into following categories:

**Objective of Analytical Study**

- Derive the analytical formulation to obtain closed-form dispersion relation of sagittal plane wave in viscoelastic multilayered periodic composites by incorporating material damping.
- Modify the viscoelastic dispersion relation for periodic viscoelastic-elastic solids and develop a systematic semi-analytical approach to perform the analysis.

**Objective of Numerical Study**

- Identify the source of fictitious modes in numerical dispersion relation of periodic layered composites.
- Provide a generalized guideline of unit cell model for numerical dispersion analysis in sagittal plane of IPMC which can avoid spatial aliasing.

**Objective of Experimental Study**

- Develop metal and polymer SHPB equipment for SEESL laboratory.
- Assemble and program electromagnetic shaker for materials and mechanics laboratory
- Determine the wave transmission behavior of periodic layered composites using vibration shaker and SHPB for small amplitude excitation.
- Evaluate the effect of wave amplitude from experimental study of layered composites using SHPB system.

1.5 Dissertation Outline

Since the dissertation focuses on systematic study of periodic composites, different aspects of the study is explained in seven chapters. Besides the Chapters related to introduction and conclusion, the contains of remaining five chapters are outlined in this section.
Chapter 2 conducts an analytical investigation for the effect of damping on sagittal plane dispersion relation of viscoelastic IPMC. The chapter provides the fundamentals of wave in viscoelastic materials and behavior of wave propagation at viscoelastic interfaces. Then, the general sagittal plane dispersion relation of viscoelastic IPMC is derived for steady state condition. Furthermore, using the proper wave transmission behavior at interface of viscoelastic and elastic solids, the chapter also modifies the dispersion relation for viscoelastic-elastic composite. Next, a periodic composite consisting of aluminum and elastomer is used to analyze the dispersion behavior for wave propagation in various angles. The complex valued modulus of the elastomer is determined from dynamic mechanical analysis (DMA) test. The result shows significant deviation caused by inherent damping of the elastomer. Finally, the analytical dispersion relation is utilized to calculate different wave motion parameters such as group slowness and transmission coefficient.

Chapter 3 provides the solution of fictitious modes in numerical dispersion relation of elastic IPMC. At first, the distinctive features of spectral distortion for wave propagation perpendicular and oblique to the layers are described. Then, the origin of fictitious modes in numerical dispersion relation is explained using spatial aliasing concept which is caused by finite sampling of FE model. By employing the analytical sagittal plane wave dispersion solution, the fictitious modes are regenerated for different wave motion angles and unit cell aspect ratios to prove the hypothesis. This chapter also provides a generalized anti-aliasing guideline using effective modulus theory to avoid fictitious modes. The efficiency of the guideline to obtain accurate numerical dispersion relation is demonstrated which remarkably concludes the long standing issue.

Chapter 4 describes the development of experimental facility of vibration shaker for small-amplitude excitation and SHPB for impact test. Especially, the complete research to establish the aluminum and nylon SHPB is explained. Next, the chapter describes frequency domain calibration of SHPB which is mandatory for the current investigation. Then, the calibration of both SHPBs are performed to identify the frequency dependent parameters of the bars. In addition, the calibration results are validated using analytical calculation. Other calibrations such as striker velocity and bar alignment are also reported in the chapter. Afterwards, the assembly of electromagnetic shaker and necessary accessories are explained. The excitation and measurement capacity of the shaker set-up is presented alongside the data acquisition system.

Chapter 5 investigates the effect of wave amplitude on transmission behavior of periodic viscoelastic composite. A combined experimental-numerical approach is employed to characterize
the amplitude-dependent property of a particular composite consisting of silicone rubber and aluminum. The frequency dependent modulus of silicone rubber is determined from DMA test for initial analytical estimation and detailed FE models to simulate experiments. The transmission spectra corresponding to small-amplitude wave is obtained using the vibration exciter. On the other hand, a hybrid SHPB system composed of aluminum input bar and nylon output bar is utilized to apply large impact load and weak transmitted signal, respectively. The wave amplitude is quantified by impulse value of the force signal. In order to extrapolate the conclusion of experimental results, a numerical SHPB model is developed and validated. The complete evolution of wave transmission spectra is studied using the numerical framework. The outcome of this study illustrates significant effect of wave amplitude on the transmission coefficient of periodic composite.
2. DISPERSION RELATION OF VISCOELASTIC MULTILAYERED COMPOSITE IN SAGITTAL PLANE

2.1 Introduction

Elastic layered composites composed of metals show unique wave characteristics typically at high frequency ranges (e.g., MHz or higher), but unwanted vibrations and noises disturbing human body are characterized by low frequency ranges (e.g., a few kHz or lower) [9, 144, 187]. In order to bring down the available frequency range of layered composites to practical ranges, researchers commonly choose polymeric materials as constituent of composites and exploit the high impedance mismatch between metals and polymers. Since experimental studies [76, 109] show that the viscoelastic behavior of polymers affects the wave transmission characteristics of layered composites, researchers have considered the viscoelasticity of polymers to study the wave characteristics of layered composites for wave propagation perpendicular to the layers. For example, using frequency-dependent complex moduli, Tanaka and Kon-No [183] analytically studied the dispersion relations of waves perpendicular to the layers in infinitely periodic layered composites composed of viscoelastic and elastic solids (i.e., viscoelastic-elastic layered composites) . Similar studies have also been performed using several numerical techniques, including finite difference method [121], variational method [122], Fourier expansion method [47], and plane wave expansion method [206]. However, there has been very little research on wave propagation at arbitrary angles in the sagittal plane of viscoelastic-elastic multilayered composites because there exist the intricate wave attenuation characteristics at the layered interfaces. In order to establish the reflection-transmission properties at viscoelastic half-space and interfaces for obliquely incident P- and SV-waves were thoroughly investigated [25, 27, 50, 102] . In addition, the special case of wave motion in elastic-viscoelastic interface [161, 175] was also studied which
requires modified formulation. This chapter addresses the challenges of the sagittal plane wave analysis in an arbitrary viscoelastic layered composite, and then focus on a class of layered composites composed of alternating viscoelastic and elastic solids (i.e., alternating viscoelastic-elastic layered composites). Note that the attenuation of harmonic plane waves occurs only in the direction perpendicular to the layers in alternating viscoelastic-elastic multilayered composites.

![Diagram](image)

**Fig. 2.1:** (A) Infinitely periodic multilayered composite in 3-D coordinate space which is periodic along $x_3$-axis. The primitive lattice vector $a_3$ defines the unit-cell of the composite with periodic length $a_3 = |a_3|$. (B) Unit cell of M-layered composite in 2-D sagittal plane where wave can propagate in any direction by making angle $\theta$ with the $x_3$-axis. Periodic length of the inclined wave is given by $(a_3 \cos \theta)$. (C) Reciprocal space of periodic composite showing the first Brillouin zone in $\kappa_1 - \kappa_3$ plane. The rectangle on right represents irreducible Brillouin zone (IBZ) which is bounded by $\kappa_3 \in [0, \pi/a_3]$ in $\kappa_3$-axis. The propagating wavevector $\kappa$ describes the wave motion at an angle $\theta$ which has periodic length $\pi/(a_3 \cos \theta)$ in the reciprocal space.

### 2.2 Phononic Crystal Background

For infinitely periodic multilayered composite (IPMC), wave motion can occur in arbitrary directions in the sagittal plane which is represented by $x_1 - x_3$ plane in Fig. 2.1A. Two types of waves can propagate in sagittal plane, namely pressure and shear waves (i.e., P- and SV-wave). For obliquely incident wave on the IPMC, the P- and SV- wave modes become coupled and the dispersion analysis should be performed in the 2-D sagittal plane. As a consequence, the analytical dispersion relation of sagittal plane wave in 1-D IPMC requires analysis in 2-D rectangular coordinate. On the other hand, the numerical dispersion relation of such composite in the FE framework require 2-D periodic structure in the $x_1 - x_3$ plane which will be explained in Chapter 3. This section briefly introduces the fundamentals of 2-D phononic crystal which are essential for the dispersion relation.
2. Dispersion Relation of Viscoelastic Multilayered Composite in Sagittal Plane

2.2.1 Bravais Lattice and Reciprocal Space

For an infinitely periodic lattice spanning in the 2-D coordinate space, the Bravais lattice of the crystal structure is described by a parallelogram shown in Fig. 2.1B. The Bravais lattice contains primitive lattice vectors $\mathbf{a}_1$ and $\mathbf{a}_3$ by which any point of the lattice can be found using the translation vector:

$$\mathbf{r} = r_1 \mathbf{a}_1 + r_3 \mathbf{a}_3$$  \hspace{1cm} (2.1)

where $r_1$ and $r_3$ are integers. Based on the primitive vectors, the smallest area named primitive unit cell can be enclosed whose periodic translations construct the complete crystal. Although the selection of primitive unit cell is not unique, the most commonly used system is the Wigner-Seitz cell. The area enclosed by a Wigner-Seitz cell is closest to the corresponding lattice point than any other point in the crystal [8]. A spatially periodic function $v(\mathbf{x})$ in any such unit cell of the crystal maintain same property when it is translated by $\mathbf{r}$, so that:

$$v(\mathbf{x} + \mathbf{r}) = v(\mathbf{x})$$  \hspace{1cm} (2.2)

where $\mathbf{x}$ is the position vector of the function. When a plane wave $e^{i\mathbf{\kappa} \cdot \mathbf{x}}$ containing wavevector $\mathbf{\kappa}$ propagates through a $\mathbf{r}$-periodic crystal, only the set of wavevector $\mathbf{g}$ that satisfies the condition:

$$e^{i\mathbf{g} \cdot (\mathbf{x} + \mathbf{r})} = e^{i\mathbf{g} \cdot \mathbf{x}}$$  \hspace{1cm} (2.3)

creates periodic plane wave $e^{i\mathbf{g} \cdot \mathbf{x}}$ in the crystal. The wavevector set $\mathbf{g}$ which is called the reciprocal lattice vector, is in fact periodic in the Fourier space of the crystal. Therefore, the spatially periodic function $v(\mathbf{x})$ can be expanded by Fourier series by:

$$v(\mathbf{x}) = \sum_{\mathbf{g}} \hat{v}_{\mathbf{g}} e^{i\mathbf{g} \cdot \mathbf{x}}$$  \hspace{1cm} (2.4)

where the Fourier coefficients $\hat{v}_{\mathbf{g}}$ are expressed as:

$$\hat{v}_{\mathbf{g}} = \frac{1}{|\Omega|} \int_{\Omega} v(\mathbf{x}) e^{-i\mathbf{g} \cdot \mathbf{x}} d\Omega$$  \hspace{1cm} (2.5)
Ω and |Ω| in (2.5) denotes a primitive unit cell and corresponding area, respectively. The reciprocal lattice g consist of primitive reciprocal vectors \( b_1 \) and \( b_3 \) defined by:

\[
\begin{align*}
    b_1 &= 2\pi \frac{\hat{e}_2 \times a_3}{\hat{e}_2 \cdot (a_3 \times a_1)}, \\
    b_3 &= 2\pi \frac{a_1 \times \hat{e}_2}{\hat{e}_2 \cdot (a_3 \times a_1)}
\end{align*}
\]  

(2.6)

where \( \hat{e}_2 = \frac{a_3 \times a_1}{\|a_3 \times a_1\|} \). Now, reciprocal lattice vectors \( g \) can be obtained as:

\[
g = g_1 b_1 + g_3 b_3
\]

(2.7)

where \( g_1 \) and \( g_3 \) are arbitrary integers. The primitive reciprocal vectors \( b_1 \) and \( b_3 \) hold orthogonal relation with the primitive lattice vectors:

\[
a_p \cdot b_q = 2\pi \delta_{pq} \quad \text{for} \quad p, q = 1, 3
\]

(2.8)

The reciprocal lattice vectors defined by (2.6) are used to obtain the Bragg planes by taking perpendicular bisection of the vectors. The area enclosed by the Bragg planes around a reciprocal lattice point is called the first Brillouin zone (BZ). Utilizing the symmetries of the first Brillouin zone, it can be reduced to the smallest reciprocal lattice space termed as irreducible Brillouin zone (IBZ) which retain all the property of the crystal. For a 1-D crystal as the IPMC, the first Brillouin zone is shown by the central rectangle in 2.1B and the smallest deeply shaded area inside the first BZ represents the irreducible Brillouin zone. If the first Brillouin zone is invariant under certain rotational or mirror reflectional transformations, it can be further reduced by all of the symmetries in the point group of the reciprocal lattice. The reduced zone is referred as the first irreducible Brillouin zone (IBZ) (see Fig. 2.1B).

### 2.2.2 Bloch Periodic Condition

Bloch theorem was discovered to solve the Schrödinger equation in the case of periodic potential. The corresponding relation can be also found for the displacement field \( u(x) \) of a \( r \)-periodic phononic crystal. When a plane wave \( e^{i\kappa x} \) propagates through the crystal, any spatial field such as displacement should satisfy the Bloch periodic condition:

\[
u(x + r) = u(x)e^{i\kappa r}
\]

(2.9)
Any wavevector $\kappa$ of the plane wave can be traced back to the first Brillouin zone using the reciprocal lattice vector $g$. As a result, it is sufficient to investigate the dispersion relation of the crystal only for a subset of the wavevector domain. Specifically, the wave frequency $\omega(\kappa)$ in the reciprocal space retains $g$-periodicity \cite{8} given by:

$$\omega(\kappa + g) = \omega(\kappa) \quad .$$

(2.10)

For elastic IPMC, destructive interference occurs at certain wave frequencies $\omega(\kappa)$ on the Bragg plane since the waves travel toward each other. This phenomenon which is termed as frequency band-gap, represents the frequency ranges at which a wave cannot travel through the phononic crystal. Due to the fact that the Bloch condition (2.9) cannot give a real solution in the bang-gap regime, the dispersion relation distinctly shows the band-gap property of the phononic crystal.

### 2.3 Oblique Wave Motion in Viscoelastic Medium

This section briefly reviews the backgrounds for wave motions in isotropic viscoelastic solids. Then, the complex Snell’s law is explained for interface between two viscoelastic solids. In addition, the law is modified to describe the transmission-reflection behavior of elastic-viscoelastic interface.

#### 2.3.1 Waves in Isotropic Viscoelastic Solid

Due to material damping, wave propagation characteristics in viscoelastic solids deviate from ones in elastic solids in many aspects \cite{23,37}. In particular, the constitutive relation for a linear viscoelastic material is often expressed in the form of the hereditary integral \cite{34}. In the absence of the body force, the equations for a homogeneous isotropic viscoelastic solid can be summarized as:

$$\nabla \cdot \sigma(x, t) = \rho \ddot{u}(x, t) \quad ,$$

$$\varepsilon(x, t) = \frac{1}{2} \left[ \nabla u(x, t) + \nabla u(x, t)^T \right] \quad ,$$

$$\sigma(x, t) = \int_{-\infty}^{t} \left[ \lambda(\tau) \text{tr}[\dot{\varepsilon}(x, t - \tau)] + 2 \mu(\tau) \dot{\varepsilon}(x, t - \tau) \right] d\tau \quad ,$$

(2.11)

where $u(x, t)$ is the displacement field at the position $x$ and the time $t$, $\sigma$ is the stress tensor, $\varepsilon$ is the strain tensor, $\mathbf{1}$ is the second order identity tensor, $\nabla$ is the nabla operator, $\text{tr}\square$ denotes
the trace, \( \rho \) is the constant mass density and \( \lambda \) and \( \mu \) are the relaxation moduli \([14]\). Note that the causality requires \( \lambda(t - \tau) = \mu(t - \tau) = 0 \) for \( \tau > t \) in (2.11)3.

Now, consider a harmonic wave motion characterized by an angular frequency \( \omega \):

\[
\mathbf{u}(x, t) = \bar{\mathbf{u}}(x)e^{i\omega t}, \quad \sigma(x, t) = \bar{\sigma}(x)e^{i\omega t}, \quad \varepsilon(x, t) = \bar{\varepsilon}(x)e^{i\omega t},
\] (2.12)

where the over-bar \( \bar{\Box} \) indicates the position-dependent amplitude. Then, the substitution of Eq. (2.12) into Eq. (2.11) provides the governing equation for the displacement field:

\[
\left[ \hat{\lambda}(\omega) + 2 \hat{\mu}(\omega) \right] \nabla^2 \bar{\mathbf{u}}(x) - \hat{\mu}(\omega) \nabla \times \nabla \times \bar{\mathbf{u}}(x) + \omega^2 \rho \bar{\mathbf{u}}(x) = 0,
\] (2.13)

where \( \hat{\lambda}(\omega) = i \omega \int_{-\infty}^{\infty} \lambda(t)e^{-i\omega t}dt \) and \( \hat{\mu}(\omega) = i \omega \int_{-\infty}^{\infty} \mu(t)e^{-i\omega t}dt \) are the complex viscoelastic moduli \([52, 64]\). Subsequently, the Helmholtz’s theorem \([26]\) can be applied to decompose the displacement field into the sum of a curl-free vector field and a divergence-free vector field using a dilatation-related scalar potential \( \bar{\Phi}(x) \) and a rotation-related vector potential \( \bar{\mathbf{H}}(x) = [\bar{H}_1 \ \bar{H}_2 \ \bar{H}_3]^T \), i.e., \( \bar{\mathbf{u}}(x) = \nabla \bar{\Phi}(x) + \nabla \times \bar{\mathbf{H}}(x) \) with \( \nabla \cdot \bar{\mathbf{H}}(x) = 0 \). Then, Eq. (2.13) becomes two decoupled governing equations:

\[
\left[ \hat{\lambda}(\omega) + 2 \hat{\mu}(\omega) \right] \nabla^2 \bar{\Phi}(x) + \omega^2 \rho \bar{\Phi}(x) = 0,
\] (2.14)

\[
\hat{\mu}(\omega) \nabla^2 \bar{\mathbf{H}}(x) + \omega^2 \rho \bar{\mathbf{H}}(x) = 0
\]

where the first equation represents the wave motion of dilatation-related excitation whereas the second one governs the rotational wave propagation in viscoelastic solids. From the above equations, the dilatational wave velocity \( c_p \) and the rotational wave velocity \( c_s \) are defined by:

\[
c_p(\omega) = \sqrt{\frac{\hat{\lambda}(\omega) + 2 \hat{\mu}(\omega)}{\rho}}, \quad c_s(\omega) = \sqrt{\frac{\hat{\mu}(\omega)}{\rho}},
\] (2.15)

where the subscripts 'p' and 's' indicate the pressure (i.e, longitudinal or dilatational) and the shear (i.e., transverse or rotational) waves, respectively. The complex-valued wave velocities imply that plane waves in viscoelastic solids are dispersive and attenuating \([52, 148]\).

Consider a plane wave characterized by a complex wavevector \( \kappa \) is propagating through the viscoelastic medium:

\[
\bar{\mathbf{u}}(x) = \bar{\mathbf{u}} e^{i\mathbf{k} \cdot x} = \bar{\mathbf{u}} e^{i\kappa \cdot x},
\] (2.16)
where $\mathbf{u}$ is the displacement amplitude along the wave plane, $\kappa$ is the wavenumber, and $\mathbf{n}$ is a unit vector indicating the direction of the plane wave. The condition for the plane wave to satisfy the governing equation (2.13) results in the following eigenvalue problem:

$$
\frac{1}{\rho\omega^2} \left[ \hat{\lambda}(\omega) + 2\hat{\mu}(\omega) \right] \mathbf{n} \otimes \mathbf{n} + \hat{\mu}(\omega) \mathbf{1} \cdot \mathbf{u} = \frac{1}{\kappa^2} \mathbf{u} ,
$$

where $\otimes$ denotes the tensor product. The non-trivial solution of wave motion $\mathbf{u}(x)$ eventually leads to two sets of complex-valued eigenvalues $\kappa$ and eigenvectors $\mathbf{u}$:

$$
\kappa_p(\omega) = \frac{\omega}{c_p(\omega)} = \sqrt{\frac{\rho\omega^2}{\hat{\lambda}(\omega) + 2\hat{\mu}(\omega)}}, \quad \kappa_s(\omega) = \frac{\omega}{c_s(\omega)} = \sqrt{\frac{\rho\omega^2}{\hat{\mu}(\omega)}},
$$

$$
\mathbf{u}_p = \mathbf{n}, \quad \mathbf{u}_s = \mathbf{m} \text{ with } \mathbf{m} \cdot \mathbf{n} = 0 ,
$$

which implies that the particle displacement of the pressure wave is parallel to the direction of the wave propagation whereas the particle displacement of the shear wave is perpendicular to the direction of the wave propagation. Here, the eigenvalues $\kappa_p(\omega)$ and $\kappa_s(\omega)$ can also be viewed as viscoelastic material properties derived from the complex viscoelastic moduli $\hat{\lambda}(\omega)$ and $\hat{\mu}(\omega)$.

In general, complex-valued wavevectors $\kappa$ can be expressed by:

$$
\kappa = \kappa^R + i\kappa^I = \hat{\kappa}^R \mathbf{n}^R + i\hat{\kappa}^I \mathbf{n}^I ,
$$

where the quantities denoted by the superscripts ‘$R$’ and ‘$I$’ are real-valued; $\mathbf{n}^Q = \kappa^Q / \|\kappa^Q\|$ for $Q = R, I$ with $\|\square\|$ denoting the Euclidean norm; and the superscript $\hat{\square}$ indicates the real-valued magnitude with respect to the corresponding wavevector. Note that $\mathbf{n}^R$ and $\mathbf{n}^I$ indicates the directions of wave propagation and wave attenuation, respectively. In addition, $\hat{\kappa}^R$ is relating to the phase of wave propagation while $\hat{\kappa}^I$ describes the amplitude of wave attenuation.

### 2.3.2 Plane Waves at Interface of Two Semi-infinite Viscoelastic Solids

Many researchers have investigated the oblique incidences of waves at viscoelastic interfaces (i.e., interface between two viscoelastic solids) [25, 27, 50, 102] as well as at viscoelastic-elastic interfaces (i.e., interface shared by elastic and viscoelastic solids) [161, 175]. This subsection briefly reviews the generalized elastodynamic Snell’s law at viscoelastic interfaces. Then, the
Snell’s law is modified to viscoelastic-elastic interfaces, which are encountered in the periodic multilayered composites composed of alternating viscoelastic and elastic solids, i.e., the focus of this study.

When a steady state plane wave encounters an interface between two viscoelastic solids, both refracted and reflected waves occur. It is also well known that the direction of wave attenuation in viscoelastic solids is generally different from the direction of wave propagation [164, 175]. Consider two viscoelastic layers, which are in contact along the plane \( x_3 = 0 \) as shown in Fig. 2.2. Then, the propagation angle \( \theta \) and the attenuation angle \( \zeta \) for each layer can be represented by \[175\]:

\[
\theta_{r,j} = \cos^{-1}\left( n^R_{r,j} \cdot e_3 \right) , \quad \zeta_{r,j} = \cos^{-1}\left( n^I_{r,j} \cdot n^R_{r,j} \right) , \quad \text{for} \quad r = p, s \quad \text{and} \quad j = 1, 2 .
\]

The attenuation angle \( \zeta \) is a unique wave characteristic of a viscoelastic solid, and it is known to be dependent on the homogeneity of a viscoelastic solid [166]. While a homogeneous plane wave is characterized by \( \zeta = 0 \), a wave with a nonzero value of \( \zeta \) is referred to as an inhomogeneous plane wave. For a plane wave encountering the considered viscoelastic layer interface, the governing equations in (2.14) resolve wave motion into two decoupled parts: anti-plane shear waves (i.e. SH-wave) and sagittal plane waves (i.e., P- and SV-waves, which are the main concern of this study). The wave motion of sagittal plane waves for the \( j \)-th layer \((j = 1, 2)\) is governed by:

\[
\nabla^2 \tilde{\Phi}_j(x) + \kappa_{p,j}^2(\omega) \tilde{\Phi}_j(x) = 0 \quad , \quad \text{for} \quad j = 1, 2 .
\]

\[
\nabla^2 \tilde{H}_{2,j}(x) + \kappa_{s,j}^2(\omega) \tilde{H}_{2,j}(x) = 0 \quad ,
\]

Subsequently, the general solution can be easily obtained in the form:

\[
\tilde{\Phi}_j(x) = \tilde{\phi}_{F,j} e^{i(\tilde{\kappa}_{p,j}^R n^R_{p,F,j} + i \tilde{\kappa}^I_{p,j} n^I_{p,F,j}) \cdot x} + \tilde{\phi}_{B,j} e^{i(\tilde{\kappa}_{p,j}^R n^R_{p,B,j} + i \tilde{\kappa}^I_{p,j} n^I_{p,B,j}) \cdot x} , \quad \text{for} \quad j = 1, 2 .
\]

\[
\tilde{H}_{2,j}(x) = \tilde{h}_{F,j} e^{i(\tilde{\kappa}_{s,j}^R n^R_{s,F,j} + i \tilde{\kappa}^I_{s,j} n^I_{s,F,j}) \cdot x} + \tilde{h}_{B,j} e^{i(\tilde{\kappa}_{s,j}^R n^R_{s,B,j} + i \tilde{\kappa}^I_{s,j} n^I_{s,B,j}) \cdot x} , \quad \text{for} \quad j = 1, 2 .
\]
where $\phi_{F,j}$, $\phi_{B,j}$, $\tilde{h}_{F,j}$, and $\tilde{h}_{B,j}$ are potential amplitudes and

$$\mathbf{n}^R_{r,F,j} = \begin{bmatrix} \sin \theta_{r,j} \\ -\cos \theta_{r,j} \end{bmatrix}, \quad \mathbf{n}^I_{r,F,j} = \begin{bmatrix} \sin(\theta_{r,j} + \zeta_{r,j}) \\ -\cos(\theta_{r,j} + \zeta_{r,j}) \end{bmatrix},$$

$$\mathbf{n}^R_{r,B,j} = \begin{bmatrix} \sin \theta_{r,j} \\ \cos \theta_{r,j} \end{bmatrix}, \quad \mathbf{n}^I_{r,B,j} = \begin{bmatrix} \sin(\theta_{r,j} + \zeta_{r,j}) \\ \cos(\theta_{r,j} + \zeta_{r,j}) \end{bmatrix},$$

for $r = p, s$. (2.24)

By substituting the general solution for $\Phi_j$ and $H_{2,j}$ into the Helmholtz’s decomposition relation for the displacement, we can obtain the sagittal plane displacement field of the $j$-th layer [26, 67]. Then, the continuous displacement boundary conditions at the layer interface (i.e., $x_3 = 0$) result in the following generalized elastodynamic Snell’s law [25–27]:

$$\hat{\kappa}^R_{p,1} \sin \theta_{p,1} = \hat{\kappa}^R_{s,1} \sin \theta_{s,1} = \hat{\kappa}^R_{p,2} \sin \theta_{p,2} = \hat{\kappa}^R_{s,2} \sin \theta_{s,2} ,$$

$$\hat{\kappa}^I_{p,1} \sin (\theta_{p,1} + \zeta_{p,1}) = \hat{\kappa}^I_{s,1} \sin (\theta_{s,1} + \zeta_{s,1}) = \hat{\kappa}^I_{p,2} \sin (\theta_{p,2} + \zeta_{p,2}) = \hat{\kappa}^I_{s,2} \sin (\theta_{s,2} + \zeta_{s,2}) ,$$

(2.25)

where the first relation is obtained from the equality of real-valued components whereas the second one is from that of imaginary-valued components.

Now, consider the special case where the first layer ($j = 1$) is elastic while the second layer ($j = 2$) is viscoelastic (see Fig. 2.3A). Due to the non-dissipative behavior of the elastic solid (i.e., $\kappa^I_{p,1} = \kappa^I_{s,1} = 0$ and see (2.20)), the wavevector relating to the first elastic layer should be real-valued, and subsequently its imaginary components in the displacement field should vanish. Consequently, the second equation in the generalized elastodynamic Snell’s law (2.25) becomes [165]:

$$\hat{\kappa}^I_{p,2} \sin (\theta_{p,2} + \zeta_{p,2}) = \hat{\kappa}^I_{s,2} \sin (\theta_{s,2} + \zeta_{s,2}) = 0 ,$$

(2.26)

which determines the attenuation angle in the second viscoelastic layer:

$$\zeta_{p,2} = -\theta_{p,2} , \quad \zeta_{s,2} = -\theta_{s,2} .$$

(2.27)

The above constraint (2.27) implies that wave attenuation in the viscoelastic layer occurs only in the direction perpendicular the layers (i.e., $\mathbf{n}^I_{p,2} \cdot e_1 = \mathbf{n}^I_{s,2} \cdot e_1 = 0$) regardless of wave propagation directions (see Fig. 2.3). In other words, while their $\kappa_3$-components remain complex,
the $\kappa_1$-component of all the wavevectors become real:

$$\kappa^I_{r,j} \cdot e_1 = \kappa^I_{r,j,1} = 0 \quad \text{for} \quad r = p, s \quad \text{and} \quad j = 1, 2 \quad ,$$

where $\kappa^I_{r,j,1}$ denotes the $\kappa_1$-component of $\kappa^I_{r,j}$. Furthermore, the same conclusion can be obtained from the case where the second layer is elastic while the first layer is viscoelastic, as shown in Fig. 2.3B. This wavevector characteristic of elastic-viscoelastic interfaces is necessary to investigate the dispersion relation of alternating elastic-viscoelastic multilayered composites in the following subsection.

![Fig. 2.2: (A) A P-wave with wavevector $\kappa_{p,1}$ (i.e. $\kappa_{p,1} = \kappa^R_{p,1} + i\kappa^I_{p,1}$) incidents at the interface between viscoelastic layers 1 and 2 with propagation angle $\theta_{p,1}$ and attenuation angle $\zeta_{p,1}$. (B-1) Reflected and transmitted P-wave denoted by subscripts 1 and 2, respectively. The propagation and attenuation vectors are represented by $\kappa^R_{p,j}$ and $\kappa^I_{p,j}$ (for $j = 1, 2$), respectively. (B-2) Illustration of reflected and transmitted SV-waves in layers 1 and 2, where propagation and attenuation wavevectors are represented by $\kappa^R_{s,j}$ and $\kappa^I_{s,j}$ (for $j = 1, 2$), respectively. The angles of propagation and attenuation waves are denoted by $\theta_{r,j}$ and $\zeta_{r,j}$ (for $j = 1, 2$ and $r = p, s)$.](image-url)
2. Dispersion Relation of Viscoelastic Multilayered Composite in Sagittal Plane

2.4 Sagittal Plane Dispersion Relation of Viscoelastic Composite

The analytical dispersion relation of elastic multilayered composites has been reported for sagittal plane waves at arbitrary directions [30, 74, 130]. As discussed in Sec. 2.3, the wave motion of viscoelastic problem is known to be similar to that of the elastic analogue problem with elastic constants replaced by their viscoelastic counterpart [52]. However, there has been very little research on sagittal plane wave dispersion relations in viscoelastic multilayered composites because of the complicacy produced by attenuation of inhomogeneous wave. The analytical
dispersion relation of viscoelastic IPMC is derived in this section and then the dispersion relation is modified for alternate elastic-viscoelastic composite. This is followed by the explanation of distinct analysis procedure for these dispersion relations.

Consider the multilayered composite of Fig. 2.1A is consists of viscoelastic layers. The considered composite has $M$ layers per unit cell, and it is infinitely periodic along $x_3$-axis. The sagittal plane for this composite is given by $x_1 - x_3$ plane, and the corresponding wavevector domain is represented by $\kappa_1 - \kappa_3$ plane, where both $\kappa_1$ and $\kappa_3$ are complex-valued. The $j$-th layer of the composite has thickness $d_j$, so that the periodic unit-cell length $a_3$ of the composite is obtained by $a_3 = \sum_{j=1}^{M} d_j$. In addition, $x_{3,j,n}$ represents the local $x_3$-coordinate for the $j$-th layer. Note that a subscript $j$ is employed to refer the characteristic properties of the $j$-th layer (e.g., $\lambda_j$, $\mu_j$, $c_{p,j}$, $c_{s,j}$, $\kappa_{p,j}$, $\kappa_{s,j}$, etc.).

The general solution of the potentials in (2.23) is employed to describe the sagittal plane wave motion of $j$-th layer. By applying this general solution to the Helmholtz’s decomposition, the displacement field of sagittal plane waves for the $j$-th layer in the $n$-th unit cell of the multilayered composite is obtained as:

$$\bar{u}_{1,j,n}(x_1, x_{3,j,n}) = \frac{\partial \Phi_j}{\partial x_1} - \frac{\partial \tilde{H}_{2,j}}{\partial x_{3,j,n}}$$

$$= \left[ - \frac{\kappa_1}{\alpha_{p,j}} P_{F,j,n} e^{-i \alpha_{p,j} x_{3,j,n}} + \frac{\kappa_1}{\alpha_{p,j}} P_{B,j,n} e^{i \alpha_{p,j} x_{3,j,n}} + Q_{F,j,n} e^{-i \alpha_{s,j} x_{3,j,n}} + Q_{B,j,n} e^{i \alpha_{s,j} x_{3,j,n}} \right] e^{i \kappa_1 x_1},$$

$$\bar{u}_{3,j,n}(x_1, x_{3,j,n}) = \frac{\partial \Phi_j}{\partial x_{3,j,n}} + \frac{\partial \tilde{H}_{2,j}}{\partial x_1}$$

$$= \left[ P_{F,j,n} e^{-i \alpha_{p,j} x_{3,j,n}} + P_{B,j,n} e^{i \alpha_{p,j} x_{3,j,n}} + \frac{\kappa_1}{\alpha_{s,j}} Q_{F,j,n} e^{-i \alpha_{s,j} x_{3,j,n}} - \frac{\kappa_1}{\alpha_{s,j}} Q_{B,j,n} e^{i \alpha_{s,j} x_{3,j,n}} \right] e^{i \kappa_1 x_1},$$

(2.29)

where $\alpha_{p,j} = \sqrt{[\kappa_{p,j}(\omega)]^2 - \kappa_1^2}$ and $\alpha_{s,j} = \sqrt{[\kappa_{s,j}(\omega)]^2 - \kappa_1^2}$ are the frequency-dependent complex coefficients relating to the propagation and attenuation angles; and $P_{F,j,n} = -i \alpha_{p,j} \tilde{\phi}_{F,j}$, $P_{B,j,n} = i \alpha_{p,j} \tilde{\phi}_{B,j}$, $Q_{F,j,n} = -i \alpha_{s,j} \tilde{h}_{F,j}$, and $P_{B,j,n} = i \alpha_{s,j} \tilde{h}_{B,j}$ are the frequency-dependent complex amplitudes to be determined from boundary conditions. Moreover, the stress field can also be
obtained for the $j$-th layer of $n$-th unit cell:

$$
\bar{\sigma}_{31,j,n}(x_1, x_3,j,n) = \hat{\mu}_j \left[ 2 \frac{\partial^2 \Phi_j}{\partial x_1 \partial x_3} - \frac{\partial^2 \bar{H}_{3,j}}{\partial x_1^2} + \frac{\partial^2 \bar{H}_{3,j}}{\partial x_3,j,n} \right] 
$$

$$
\bar{\sigma}_{33,j,n}(x_1, x_3,j,n) = \left( \hat{\lambda}_j + 2\hat{\mu}_j \right) \left( \frac{\partial^2 \Phi_j}{\partial x_1^2} + \frac{\partial^2 \Phi_j}{\partial x_3,j,n} \right) - 2\hat{\mu}_j \left( \frac{\partial^2 \Phi_j}{\partial x_1 \partial x_3,j,n} \right)$$

$$
= \left[ -P_{F,j,n} \left( \hat{\lambda}_j + 2\hat{\mu}_j \right) \frac{\alpha_{p,j}^2}{\alpha_{s,j}} e^{-i\alpha_{s,j}x_{3,j,n}} + P_{B,j,n} \left( \hat{\lambda}_j + 2\hat{\mu}_j \right) \frac{\alpha_{p,j}^2 + \hat{\lambda}_j \kappa_1^2}{\alpha_{p,j}} e^{i\alpha_{s,j}x_{3,j,n}} - 2Q_{F,j,n} \hat{\mu}_j \kappa_1 e^{-i\alpha_{s,j}x_{3,j,n}} - 2Q_{B,j,n} \hat{\mu}_j \kappa_1 e^{i\alpha_{s,j}x_{3,j,n}} \right] e^{i(\kappa_1 x_1 - \omega t)} .
$$

(2.30)

Here, for the $j$-th layer in the $n$-th unit-cell, the complex-valued amplitude vector $W_{j,n}$ is found as:

$$
W_{j,n} = \begin{bmatrix} P_{F,j,n} \\ P_{B,j,n} \\ Q_{F,j,n} \\ Q_{B,j,n} \end{bmatrix} .
$$

(2.31)

Then, continuous displacement and stress boundary conditions (5.9) are employed at the interface between $j$-th and $(j + 1)$-th unit cells. Upon successive application of such boundary conditions to all the layer interfaces in $n$-th unit-cell results in the following relation between the adjacent unit-cells [74]:

$$
W_{1,n+1} = TW_{1,n}
$$

(2.32)

where

$$
T = T(\omega, \kappa_\theta, \theta, \zeta) = R_1^{-1} \left( \prod_{j=2}^{M-1} R_{M+2-j}^M D_{M+2-j} R_{M+2-j}^{-1} \right) R_1 D_1 .
$$

(2.33)
2. Dispersion Relation of Viscoelastic Multilayered Composite in Sagittal Plane

\[ R_j = \begin{bmatrix} 
\frac{1}{\alpha_{p,j}} & \frac{1}{\alpha_{s,j}} & \frac{\kappa_1}{\alpha_{s,j}} & -\frac{\kappa_1}{\alpha_{s,j}} \\
\frac{(\lambda_j+2\mu_j)\alpha_{p,j}}{\alpha_{p,j}} & \frac{(\lambda_j+2\mu_j)\alpha_{s,j}}{\alpha_{p,j}} & -2\mu_j\kappa_1 & -2\mu_j\kappa_1 \\
-\frac{\kappa_1}{\alpha_{p,j}} & \frac{\kappa_1}{\alpha_{p,j}} & \frac{\mu_j(\kappa_2^2-\alpha_{s,j}^2)}{\alpha_{s,j}} & -\frac{\mu_j(\kappa_2^2-\alpha_{s,j}^2)}{\alpha_{s,j}} \\
2\mu_j\kappa_1 & 2\mu_j\kappa_1 & 1 & 1 
\end{bmatrix}, \tag{2.34} \]

\[ D_j = \begin{bmatrix} 
e^{-i\alpha_{p,j}d_j} & 0 & 0 & 0 \\
0 & e^{i\alpha_{p,j}d_j} & 0 & 0 \\
0 & 0 & e^{-i\alpha_{s,j}d_j} & 0 \\
0 & 0 & 0 & e^{i\alpha_{s,j}d_j} 
\end{bmatrix}, \tag{2.35} \]

where \( T \) is the transfer matrix establishing the relation between the complex-valued amplitude vectors of adjacent unit-cells in the sagittal plane. In addition, the Bloch periodic condition provides the second relation between analogous amplitude vectors of adjacent unit-cells [8]:

\[ W_{1,n+1} = e^{i\kappa_3 a_3} W_{1,n}. \tag{2.36} \]

Now, the substitution of the Bloch periodic condition (2.36) to (2.32) provides an eigenvalue problem:

\[ TW_{1,n} = e^{i\kappa_3 a_3} W_{1,n}. \tag{2.37} \]

where \( e^{i\kappa_3 a_3} \) and \( W_{1,n} \) are the eigenvalue and the eigenvector of the transfer matrix \( T \), respectively. By setting \( e^{i\kappa_3 a_3} = \eta \), Cayley-Hamilton theorem [20] provides the fourth-order characteristic polynomial of the transfer matrix \( T \):

\[ \eta^4 - g_3\eta^3 + g_2\eta^2 - g_1\eta + g_0 = 0 \tag{2.38} \]

where

\[ g_3(\omega, \kappa_\theta, \theta, \zeta) = g_1(\omega, \kappa_\theta, \theta, \zeta) = \text{tr}(T), \]

\[ g_2(\omega, \kappa_\theta, \theta, \zeta) = \frac{1}{2} \left[ (\text{tr}(T))^2 - \text{tr}(T^2) \right], \tag{2.39} \]

\[ g_0(\omega, \kappa_\theta, \theta, \zeta) = 1. \]

where \( \text{tr}(\square) \) and \( \text{det}(\square) \) denote the trace and the determinant of a matrix, respectively. Here, the polynomial coefficients \( g_0, \ldots, g_3 \) can also be determined from four roots \( \eta_r = e^{i\kappa_3 a_3} \) (with \( r = 1, \ldots, 4 \)) of (2.38). Recall that the dispersion relation is symmetric along \( \kappa_1 \)- and \( \kappa_3 \)-axes.
due to the symmetry of geometric configuration of infinitely periodic multilayered composites. Thus, for a given $\kappa_1 \in [0, \infty)$, only two out of four solutions provide distinctively different eigenmodes (denoted by $\kappa_{3,1}$ and $\kappa_{3,2}$), while the other two solutions are located at the symmetric position along $\kappa_3$-axis, i.e., $\kappa_{3,3} = -\kappa_{3,1}$ and $\kappa_{3,4} = -\kappa_{3,2}$. Consequently, the roots of the characteristic polynomial (2.38) retain the reciprocal relation, i.e., $\eta_3 = 1/\eta_1$ and $\eta_4 = 1/\eta_3$.

Now, the polynomial coefficients $g_1(\omega, \kappa_1)$ and $g_0(\omega, \kappa_1)$ become simplified:

$$g_1(\omega, \kappa_1) = g_3(\omega, \kappa_1) = \text{tr}(T_s)$$
$$g_0(\omega, \kappa_1) = 1.$$  \hfill (2.40)

The simplified values of coefficients are replaced in the characteristics polynomial equation (2.38) to obtain the dispersion relation of viscoelastic IPMC as:

$$\cos (\kappa_3 a_3) = \frac{1}{4} \left[ g_3(\omega, \kappa_1) \pm \sqrt{[g_3(\omega, \kappa_1)]^2 - 4g_2(\omega, \kappa_1) + 8} \right]. \hfill (2.41)$$

However, when the viscoelastic layer is attached to an elastic layer, (2.28) indicates that wave attenuation in a viscoelastic layer occurs only in the direction perpendicular the layers, i.e., zero $\kappa_1$-component of $\kappa^I$. Thus, this wave propagation characteristic (i.e., real-valued $\kappa_1$-component) enables in direct application of the analytical approach employed in elastic multilayered composites to the dispersion relation of alternating viscoelastic-elastic multilayered composites. Nevertheless, the wavenumber component along $\kappa_3$-axis which is part of the argument of cosine in (2.41), is complex-valued. Under such condition, the dispersion relation (2.41) can be solved for real-valued $\kappa_1$ and complex-valued $\kappa_3$. Therefore, the invariants given by (2.39) becomes functions of $\omega$ and $\kappa_1$. Taking consideration of this wave motion behavior, the dispersion relation (2.41) can be modified for elastic-viscoelastic periodic multilayered composites as:

$$\cos (\kappa_3 a_3) = \frac{1}{4} \left[ g_3(\omega, \kappa_1) \pm \sqrt{[g_3(\omega, \kappa_1)]^2 - 4g_2(\omega, \kappa_1) + 8} \right]. \hfill (2.42)$$

When an alternate viscoelastic-elastic composite is analyzed for dispersion, real-valued wavenumber $\kappa_1 \in [0, \infty)$ can be employed as an input alongside the angular frequency $\omega \in [0, \infty)$. The dispersion analysis scheme of such composite is explained in Fig. 2.4, where the propagation and the attenuation characteristics of sagittal plane waves can be formally presented in the two 2-D plots of $\omega$ over the $\kappa_1^R - \kappa_3^R$ plane and $\omega$ over the $\kappa_1^I - \kappa_3^I$ plane, respectively. For wave
propagation at angle $\theta$, the complex valued $\kappa_\theta = \kappa_3^R + i\kappa_3^I$ is obtained for real-valued set ($\kappa_1$, $\omega$) from (2.42). The real part of the solution (i.e., $\kappa_3^R$) can be traced to the $\kappa_\theta^R - f$ plane by $\kappa_\theta^R = \kappa_3^R / \cos \theta$. Due to attenuation only along $\kappa_3^I$-axes, the corresponding imaginary solution lies on the $\kappa_3^I - f$ plane. This solution is illustrated in Fig. 2.4 by three distinct points which shows that for real component $\kappa_\theta^R$, the associated imaginary component $\kappa_3^I$ is found from projection on the $\kappa_3^I - f$ plane.

**Fig. 2.4:** Three dimensional scheme of $f - \kappa_1 - \kappa_3$ coordinate system to determine dispersion relation of viscoelastic-elastic composite. The phase dispersion relation is shown by solid line in the wave propagation plane inclined at angle $\theta$ on the $\kappa_1 - \kappa_\theta^R$ coordinate plane. The inclined length of the phase dispersion plane is $\pi/a_3 \cos \theta$ which has projection length of $\pi/a_3$ on the $\kappa_3^R$-axis. Three distinct points of $\kappa_\theta^R - f$ dispersion relation is projected on $\kappa_3^I - f$ plane which represent corresponding attenuation relation.

### 2.5 Dispersion Analysis Results

In this section, a periodic bilayered composite composed of alternating a metal and a viscoelastic polymer is considered to investigate the dispersion relation of sagittal plane waves at four different angles $\theta = 0^\circ$, $15^\circ$, $30^\circ$, and $60^\circ$. In order to illustrate the distinct effects of viscoelastic properties, an elastic counterpart of the alternating polymer-metal composite is also analyzed, where only the static elastic properties of the polymer is utilized by neglecting its damping properties. In addition to the dispersion relation, the transmission coefficient and the group slowness are calculated from the complex-valued dispersion relations.
2. Dispersion Relation of Viscoelastic Multilayered Composite in Sagittal Plane

2.5.1 Geometry and Materials of Viscoelastic-Elastic Composite

The dispersion relation of sagittal plane waves is studied using a bilayered composite composed of alternating aluminum (aluminum 6061-T6) and polyurethane elastomer (Hapflex-560 from Hapco Inc.). The subscripts 1 and 2 used for describing material and geometric properties, denote the aluminum layer and the polyurethane layer, respectively. The unit-cell of the composite comprises an aluminum layer of $d_1 = 10 \text{ mm}$ and a polyurethane layer of $d_2 = 10 \text{ mm}$, so that the unit-cell length along the $x_3$-axis is $a_3 = d_1 + d_2 = 20 \text{ mm}$.

For the material properties of aluminum, mass density of $\rho_1 = 2700 \text{ kg/m}^3$ and the frequency-independent elastic moduli of $\lambda_1 = 51.1 \text{ GPa}$ and $\mu_1 = 26.3 \text{ GPa}$ are used. On the other hand, the complex-valued viscoelastic moduli of the considered polyurethane are experimentally determined by performing dynamic mechanical analysis (DMA) using RSA-G2 Solids Analyzer from TA Instruments. Two different samples were tested, and the deviation was very small, showing only around 1% for both storage and loss modulus of polyurethane elastomer. Experiments are performed to obtain the complex-valued tensile modulus $\hat{\lambda}_2(\omega) = \hat{E}_2(\omega) + i \hat{\mu}_2(\omega)$ and shear modulus $\hat{\mu}_2(\omega) = \hat{\mu}_2'(\omega) + i \hat{\mu}_2''(\omega)$, where $\square'$ and $\square''$ denote the storage modulus and loss modulus, respectively. Then, $\hat{\lambda}_2(\omega)$ is determined through the relation, $\hat{\lambda}_2 = \frac{\hat{\mu}_2(E_2-2\hat{\mu}_2)}{3\hat{\mu}_2-E_2}$. The experimentally obtained complex moduli (i.e., $\hat{\lambda}_2(\omega)$ and $\hat{\mu}_2(\omega)$) of polyurethane are presented in Fig. 5.1, where the linear frequency denoted by $f = \omega/(2\pi)$ is employed instead of the angular frequency $\omega$. The mass density of the considered polyurethane is $\rho_2 = 1060 \text{ kg/m}^3$.

In order to highlight the effects of viscoelastic properties in the dispersion analysis of sagittal plane waves, two sets of material models for the considered composite are explored: one set with elastic aluminum and viscoelastic polyurethane models and another with elastic aluminum and pseudo-elastic polyurethane models. Here, for pseudo-elastic polyurethane model, we extract the frequency-independent elastic moduli (i.e., $\hat{\lambda}_2^{pe} = 41.5 \text{ MPa}$ and $\hat{\mu}_1^{pe} = 6.76 \text{ MPa}$ and see the dotted lines in Fig. 5.1) from its storage moduli at the zero frequency ($f = 0$), which can be viewed as a long-term static behavior.

2.5.2 Complex-valued Dispersion Relations

For the considered periodic bilayered composite composed of alternating polyurethane and aluminum, the analytical expression of the dispersion relation (2.42) can provide the complete
2. Dispersion Relation of Viscoelastic Multilayered Composite in Sagittal Plane

Fig. 2.5: Viscoelastic properties of polyurethane elastomer obtained by dynamic mechanical analysis (DMA). (A) frequency dependent modulus $\hat{\lambda}(\omega)$. (B) frequency dependent modulus $\hat{\mu}(\omega)$. The storage and loss moduli are represented by circle- and square-marked solid lines, respectively. Note that the shaded areas denote one standard deviation from two DMA tests. The constant modulus of pseudo-elastic approximation is shown by dotted lines.

dispersion relation of the sagittal plane in the complex wavevector domain. In this study, the effects of viscoelasticity on sagittal plane waves are investigated by considering harmonic plane waves at four different incident angles, $\theta = 0^\circ$, 15$^\circ$, 30$^\circ$, and 60$^\circ$, where $\theta = \cos^{-1}\left(\frac{\kappa_R}{\|\kappa_R\|} \cdot e_3\right)$ (see Fig. 2.1). Here, plane waves attenuate only in the direction perpendicular the layers (i.e., $\kappa_I \cdot e_1 = 0$ and see Fig. 2.3), so the propagation and the attenuation dispersion relations are presented in $f - \kappa^R_\theta$ plane and $f - \kappa^I_3$ plane, respectively. Note that $\kappa^R_\theta$ denotes the wavenumber in the direction of the wave propagation angle $\theta$, and the representation scheme is illustrated in Fig. 2.4. The real component $\kappa^R_\theta$ is obtained from identifying the intersections of the $\theta$-plane and the collection of the calculated $\kappa^R_3$. Accordingly, the associated imaginary component $\kappa^I_3$ is found and is projected on the $\kappa^I_3 - f$ plane. As a result, more than two wave modes can exist for a given frequency at arbitrary oblique wave motion (see Figs. 2.6 and 2.7).

Fig. 2.6 presents the dispersion relations of sagittal plane waves in the alternating polyurethane-aluminum bilayered composite, where polyurethane is modeled as a pseudo-elastic material model using elastic properties extracted from a long-term static behavior. Similarly, Fig. 2.7 shows the corresponding dispersion relations of the considered bilayered composite, where polyurethane is modeled as a viscoelastic material model using the measured complex moduli shown in Fig. 5.1. While the left columns of Figs. 2.6 and 2.7 show the wave attenuation characteristics represented in the $\kappa^I_3 - f$ space, the right columns present the dispersion relation represented in the $\kappa^R_\theta - f$ space. Due to the application of the Bloch periodic boundary condition (2.36), the real part of wavenumber $\kappa^R_\theta$ (i.e., propagation characteristics) is confined within the IBZ.
2. Dispersion Relation of Viscoelastic Multilayered Composite in Sagittal Plane

(i.e., $\kappa^R_y \in [0, \pi/(a_3 \cos \theta)]$) as shown in the right columns of Figs. 2.6 and 2.7. However, the left column shows that the imaginary part of wavenumber $\kappa^I_3$ is unbounded (i.e., $\kappa^I_3 \in [0, \infty)$). This is due to the fact that wave attenuation is independent of Bloch periodicity.

2.5.3 Group Slowness and Transmission Coefficient

In order to qualitatively investigate the wave transmission characteristics in the considered composite, the group slowness and the transmission coefficient are calculated from the obtained dispersion relation. The group slowness $S_g$ is defined as the inverse of group velocity, and it measures the rate of change in wavenumber with respect to angular frequency:

$$S_g = \frac{d\kappa^R_y}{d\omega}.$$  \hspace{1cm} (2.43)

The group slowness has been adopted to investigate wave directionality in iso-frequency surfaces [35, 36], locally-resonant characteristics [184, 203], and sound absorption [68, 69]. Since its definition is closely related to the density of states, the group slowness also describes the number of wave modes per unit frequency range. In other words, high group slowness values at a specific frequency imply large density of wave modes at the considered frequency range. In this study, the group slowness is calculated from the real part of the wavenumber $\kappa_3$ (i.e., recall $\kappa^R_y = \kappa^R_3 / \cos \theta$), and it is presented in the right columns of Figs. 2.8 and 2.9 for the pseudoelastic-elastic and the viscoelastic-elastic composite, respectively. However, the group slowness property of the composite does not convey the wave transmission behavior.

Therefore, in addition to the group slowness which is governed by the phase of waves, the transmission coefficient from the imaginary part of the wavenumber $\kappa_3$ is also computed. The transmission coefficient is commonly used to investigate the attenuation characteristics of periodic layered composites [28, 32, 58, 92, 153, 168, 169, 174]. Recall that the Bloch periodic condition can also be established between $n$-th and $(n + N)$-th unit-cell amplitude vectors:

$$W_{1,n+N} = e^{i\kappa_3 N a_3} W_{1,n},$$  \hspace{1cm} (2.44)

where $N a_3$ denotes the distance between the two considered unit-cells. In this study, the transmission coefficient is defined by taking the displacement amplitude ratio between an incident wave and the corresponding transmitted wave. By substituting $\kappa_3 = \kappa^R_3 + i \kappa^I_3$, the transmission
Fig. 2.6: Complete dispersion analysis results of pseudo-elastic composite showing attenuation relations $\kappa^I - f$ for wave motions at (A-1) $\theta = 0^\circ$, (B-1) $\theta = 15^\circ$, (C-1) $\theta = 30^\circ$ and (D-1) $\theta = 60^\circ$. The phase dispersion relations $\kappa^R - f$ are illustrated for wave motions at (A-2) $\theta = 0^\circ$, (B-2) $\theta = 15^\circ$, (C-2) $\theta = 30^\circ$ and (D-2) $\theta = 60^\circ$ where the wavevector $\kappa^R \in [0, \pi/(a_3 \cos \theta)]$ inside the IBZ varies with propagation angles.
Fig. 2.7: Complete dispersion analysis results of viscoelastic-elastic composite showing attenuation relations $\kappa^I - f$ for wave motions at (A-1) $\theta = 0^\circ$, (B-1) $\theta = 15^\circ$, (C-1) $\theta = 30^\circ$ and (D-1) $\theta = 60^\circ$. The phase dispersion relations $\kappa^R - f$ are illustrated for wave motions at (A-2) $\theta = 0^\circ$, (B-2) $\theta = 15^\circ$, (C-2) $\theta = 30^\circ$ and (D-2) $\theta = 60^\circ$ where the wavevector $\kappa^R \in [0, \pi/(a_3 \cos \theta)]$ inside the IBZ varies with propagation angles.
coefficient $C_t$ is defined by:

$$C_t = \frac{\|W_{1,n+N}\|}{\|W_{1,n}\|} = e^{-\kappa_3^f N a_3},$$

(2.45)

where the Euclidean norm $\|\| \|$ is adopted for the magnitude of the complex-valued amplitude vectors. Note that the transmission coefficient becomes unity (i.e., $C_t = 1$) as $\kappa_3^f$ decreases to zero (i.e., $\kappa_3^f = 0$) implying propagation of wave without any attenuation. On the other hand, the transmission coefficient converges to zero (i.e., $C_t = 0$) which suggest complete attenuation of wave. Unlike in elastic layered composites, the magnitude of the transmission coefficient of viscoelastic-elastic layered composites is significantly affected by $N$. In order to attain the phononic crystal characteristics form periodic composites, multiple number of unit cells should be considered. Although, the number of unit cells cannot be infinite for practical purpose. Note that, transmission coefficient $C_t \to 0$ for almost entire frequency range when $N \to \infty$, which is rather impractical. Therefore, this study have selected $N = 3$ to directly calculate the transmission coefficients from $\kappa_2^f$ with $N = 3$ for both types of composite. The left columns of Figs. 2.8 and 2.9 show the transmission coefficients of the pseudo-elastic material model and the viscoelastic material model, respectively. When it comes to the interpretation of the attenuation characteristics of waves, a transmission coefficient plot is more practical than the attenuation plot of $\kappa_3^f - f$ because an experimentally measured transfer function can be directly comparable to the corresponding transmission coefficient plot.

### 2.6 Discussion

At oblique incident waves, Fig. 2.6 and 2.7 show the multi-modal behavior (i.e., more than two modes) in $\kappa_0^R - f$ plots and spiral intersections in $\kappa_3^f - f$ plots. As described in Sec. 2.5.2, this phenomenon is indebted to the 2-dimensional representation (i.e., $\kappa_0^R - f$ and $\kappa_3^f - f$) of the complex-valued dispersion relation (i.e., relation between $\kappa = [\kappa_1 \ \kappa_3^R]^T + i [0 \ \kappa_3^f]^T$ and $f$) of the considered layered composite. According to the illustration in Fig. 2.4, the results shown in Fig. 2.6 and 2.7 can be viewed as the projection of the complex-valued dispersion relation onto the $\kappa_0^R - f$ and $\kappa_3^f - f$ planes for a given frequency value.

Wave propagation perpendicular to the layers is characterized by $\theta = 0$ (consequently, $\kappa_1 = 0$; see Fig. 2.4), and it contains only two uncoupled wave modes at any given frequency as shown in Fig. 2.6A-1 and 2.7A-1. For wave motion at oblique angles with $\theta > 0$, the wavenumber $\kappa_1$ varies within the Brillouin zone (i.e., $0 \leq \kappa_1 \leq (\pi/a_3) \tan \theta$), and (2.42) provides a pair of
Fig. 2.8: Transmission coefficient $C_t$ of periodic pseudo-elastic composite calculated from (5.11) for wave motion at (A-1) $\theta = 0^\circ$, (B-1) $\theta = 15^\circ$, (C-1) $\theta = 30^\circ$ and (D-1) $\theta = 60^\circ$. Group slowness $S_g$ obtained using (2.43) for wave motion at (A-2) $\theta = 0^\circ$, (B-2) $\theta = 15^\circ$, (C-2) $\theta = 30^\circ$ and (D-2) $\theta = 60^\circ$. 
Fig. 2.9: Transmission coefficient $C_t$ of periodic viscoelastic-elastic composite calculated from (5.11) for wave motion at (A-1) $\theta = 0^\circ$, (B-1) $\theta = 15^\circ$, (C-1) $\theta = 30^\circ$ and (D-1) $\theta = 60^\circ$. Group slowness $S_g$ obtained using (2.43) for wave motion at (A-2) $\theta = 0^\circ$, (B-2) $\theta = 15^\circ$, (C-2) $\theta = 30^\circ$ and (D-2) $\theta = 60^\circ$. 
eigenvalues possessing coupled wave modes (see [74]). Note that, the dispersion relations in Fig.
2.6 and 2.7 are obtained by intersecting the oblique $\kappa_\theta^R - f$ plane with the continuous evolution
of the eigenvalue pair on the $\kappa_1 - f$ plane. Thus, the identified dispersion relation may result
in more than two coupled modes on the projected $\kappa_\theta^R - f$ plane at a given frequency and may
engender spiral intersections on the $\kappa_3^I - f$ plane (see Fig. 2.7B, 2.7C, and 2.7D). It is worth
mentioning that multi-modal behavior in the $\kappa_\theta^R - f$ plane is accompanied by spiral intersections
in the $\kappa_3^I - f$ plane. A representative example can be found near $f \approx 9 kHz$. A reference [182]
has also reported a similar phenomenon for wave motion at oblique angles in infinitely periodic
elastic layered composites.

From the identified complex-valued dispersion relation from (2.42) (i.e. results in Fig. 2.6 and
2.7), the group slowness $S_g$ and transmission coefficient $C_t$ (see Fig. 2.8 and 2.9) are analytically
calculated, which are frequently adopted to evaluate qualitatively the wave characteristics of
phononic crystals. The physical meaning of the group slowness (see (2.43)) resembles that of
the density of states, which measures the number of states (or wave modes) at each frequency.
In other words, high group slowness values at a specific frequency imply an abundance of wave
modes in the considered media. The high peaks in the group slowness plots (Fig. 2.8 and 2.9)
correspond to nearly horizontal dispersion relation (i.e., $\kappa_\theta^R - f$ plot) in Fig. 2.6 and 2.7. On
the other hand, the zero group slowness value indicates absolutely no wave propagation through
the medium, and it clearly identifies the complete phononic band-gaps for the pseudo-elastic
composite at the incident angle $\theta = 0^o$ (see Fig. 2.6A-2 and 2.8A-2). For the viscoelastic-elastic
layered composite, however, the group slowness investigation shows that there is no complete
phononic band-gaps regardless of the incident angle due to the simultaneous wave propagation
and attenuation in the medium (see Fig. 2.8: right column). Note that the behavior of the
group slowness is highly localized along the frequency because it is defined by the derivatives of
wavenumber with respect to frequency.

While the group slowness $S_g$ focuses on wave modes in medium, the transmission coefficient
$C_t$ defined in (5.11) measures amplitude attenuation. A high transmission coefficient indicates
a large wave attenuation. In the pseudo-elastic layered composite, the maximum transmission
coefficient value is predominantly one at oblique incident angles (see Fig. 2.8: left column).
However, the transmission coefficient plots for the viscoelastic-elastic layered composite show
a large wave attenuation only near a low frequency range (i.e., below 10 kHz). Furthermore,
the spiral spiral intersections observed in the $\kappa_3^I - f$ plot do not necessarily affect the wave
attenuation behavior because the definition of the transmission coefficient is dependent on the magnitude of $\kappa_3^I$.

In the dispersion relation of pseudo-elastic layered composites, the wavenumber $\kappa_3$ can be either pure real number, pure imaginary number, or complex conjugate pair, which represent propagating band, attenuating band, and anticrossing band induced from the mixed mode interaction, respectively [39, 83, 118, 159, 181, 182]. For wave propagation perpendicular to the layers (i.e., $\theta = 0^\circ$), the dispersion relation in Figs. 2.6A display several complete stop bands (e.g., around $5 \text{kHz}$, $7 \text{kHz}$, $12 \text{kHz}$, $15 \text{kHz}$, etc.), which are characterized by non-zero $\kappa_3^I$ with an infinite group velocity $df/d\kappa_3^R$. Furthermore, the existence of those stop bands are also confirmed from the transmission coefficient and the group slowness in Figs. 2.8A. Note that vertical lines on the right edge of the dispersion relations $\kappa_3^R - f$ in the right columns of Fig. 2.6 represent a stop band for the considered wave mode, indicating $df/d\kappa_3^R = \infty$. However, all those complete stop bands disappear at oblique angles. The transmission coefficient of unity in Figs. 2.8B/D shows that there exist at least one propagating wave mode within the considered frequency ranges (i.e., up to $20 \text{kHz}$). In Fig. 2.8B/C/D from the pseudo-elastic material model, vertical lines in the transmission coefficient (i.e., $C_t = 1$) represent propagating wave modes with $\kappa_3^I = 0$. In other words, the transmission coefficient of unity in Fig. 2.8B/C/D shows that there exist at least one propagating wave mode within the considered frequency ranges (i.e., up to $20 \text{kHz}$).

On the other hand, Fig. 2.7 shows the dispersion relation of sagittal plane waves in the alternating polyurethane-aluminum bilayered composite, where polyurethane is modeled as a viscoelastic material using the measured complex moduli shown in Fig. 5.1. The wavenumber $\kappa_3$ in the dispersion relation of viscoelastic-elastic composite is always complex-valued due to the presence of inherent material damping of viscoelastic layers, implying that there are neither complete stop bands nor propagation bands without any attenuation. In other words, the dispersion relations of Fig. 2.7 illustrates that all the modes of sagittal plane waves in viscoelastic-elastic layered composites are simultaneously propagating and attenuating. No complete band-gap is observed even for wave propagation perpendicular to the layers [126, 183], but Figs. 2.7 and 2.9 indicate that there exist some low wave transmission regions. For instance, Figs. 2.9A-1 shows that frequency contents above $7 \text{kHz}$ at $\theta = 0^\circ$ substantially attenuate within a distance of 3 unit-cells (i.e., $N = 3$). This wave motion in viscoelastic-elastic layered composites is distinctively different from that in the elastic counterpart composites, whose
2. Dispersion Relation of Viscoelastic Multilayered Composite in Sagittal Plane

The similar trends can also be observed in the cases of different incident angles shown in the left column of Fig. 2.9. Note that the largest envelope of a transmission coefficient plot is determined by the smallest $\kappa_3^I$ envelope due to the definition of the transmission coefficient in (5.11). Thus, the left column of Fig. 2.9 implies that the use of viscoelastic layers can prevent high frequency wave motions (e.g., $> 15 \text{ kHz}$), but there is not much attenuation in low frequency wave motions (e.g., $< 7 \text{ kHz}$) regardless of wave propagation directions. Fig. 2.9 clearly shows that the viscoelastic material model reveals the distinctive wave attenuation, which cannot be captured by the pseudo-elastic material model (see Fig. 2.8).

In addition, there is another noteworthy feature in the phase dispersion relation $\kappa_0^R - f$ of a viscoelastic-elastic layered composite, compared to that of the elastic counterpart. The right column of Fig. 2.7 shows that wave modes in viscoelastic-elastic layered composite are placed in the higher frequency ranges than those in the pseudo-elastic model, and this is due to the employment of the frequency-dependent moduli. Fig. 5.1 exhibits that the frequency-dependent viscoelastic storage moduli of polyurethane is getting stiffer as frequency increases, whereas the pseudo-elastic material model has the frequency-independent elastic moduli. The most remarkable outcome of this study is that, low wave transmission can be achieved for oblique wave motion in periodic composite by using the accurate viscoelastic behavior of the polymer. In other words, while pseudo-elastic composite exhibits full transmission for oblique wave motion in Figs. 2.8B/C/D, the actual viscoelastic-elastic composite reveals the significant wave attenuation in Figs. 2.9B/C/D.

2.7 Conclusion

The general analytical formulation is employed to obtain the dispersion relations of viscoelastic IPMC. In addition, for alternating viscoelastic-elastic layered composites the attenuation of harmonic plane waves is found to occur only in the direction perpendicular to the layers. By using this wave propagation characteristic, the semi-analytical approach required for elastic multilayered composites, is directly employed to find the dispersion relation of sagittal plane waves in alternating viscoelastic-elastic multilayered composites. A specific bilayered composite composed of alternating aluminum and polyurethane elastomer is considered for analysis, whose complex-valued viscoelastic moduli are experimentally determined by performing DMA. In order to illustrate the distinct effects of its viscoelastic properties, the wave motion of the alternating
viscoelastic-elastic layered composite is compared with that of its elastic counterpart, where the frequency-independent elastic moduli of polyurethane elastomer are extracted from its storage moduli at the zero frequency. The analysis shows that the alternating viscoelastic-elastic layered composite does not possess a phononic band-gap, regardless of incident angles. In addition, wave motions at oblique angles (other than $\theta = 0^\circ$) are found to travel with a wide range of frequency contents. The presented analysis demonstrates that wave dispersion relation in viscoelastic-elastic layered composites is distinctively different from the corresponding elastic counterpart, and highlights the importance of the viscoelastic modeling of polymeric materials in wave dispersion analysis. Furthermore, the complex-valued dispersion relation is utilized to obtain the transmission coefficient and group slowness of both viscoelastic-elastic and pseudo-elastic composites. While the pseudo-elastic composite exhibit full transmission behavior for all oblique wave motions, the actual viscoelastic-elastic composite is characterized with significant wave attenuation in most frequency ranges. On the other hand, the group slowness property of periodic viscoelastic-elastic composite manifest the localized slowness spike similar to the elastic counterpart which can be exploited for various applications.
3. SPATIAL ALIASING SOLUTION OF NUMERICAL DISPERSION RELATION

In order to implement the dispersion relations of IPMC to intricate applications, researchers have been employing various numerical techniques for dispersion analysis. In particular, the FE method offers a remarkable framework to efficiently investigate the effect of material and geometric nonlinearity to phononic dispersion relations [15, 120, 171, 189, 190], which can be hardly done using other numerical techniques. So, it has become a prevalent method to study the evolution of phononic dispersion relations due to the effect of nonlinearities of structures. In the FE method, a two-step manner is typically employed to study the evolution of dispersion relation of a periodic composite subject to external loading. The first step is a conventional static nonlinear analysis on the periodic composite, and this step considers material and geometric nonlinearities in the FE framework. In the second step, a linear perturbation analysis is performed on the deformed periodic composite to obtain dispersion relations. Through this procedure, researchers in the FE community have opened the possibility of tuning the band-structure of periodic structures [145, 171, 189, 190, 200].

However, despite its superior capability to investigate the effect of nonlinearities to phononic dispersion relations, the FE method suffers from spectral distortions in the dispersion analysis of waves motion in layered composites [2, 70, 117, 127]. It is worth mentioning that the spectral distortion discussed in those references is irrelevant to the effect of mesh size in FE models. In order to avoid the spectral distortion, only phenomenological approach is suggested to be employed. However, this issue has never been thoroughly investigated, and there has been no specific guideline for a proper unit cell configuration for numerical dispersion relations. Specifically, the structural 1-D elements (e.g., truss or beam elements) available in the FE method are not suited for the dispersion analysis of infinitely periodic layered composites because an infinite length in the direction parallel to the layers (i.e., $x_1$ direction in Fig. 2.1B) cannot be considered.
in such elements. Thus, researchers in the FE community have to adopt 2-D elements and applied the redundant Bloch periodic condition along the direction parallel to the layers [2, 70, 117, 127]. The spectral distortion of the numerical dispersion relation is systematically investigated in this chapter. The origin of fictitious modes for wave propagation perpendicular and oblique to the layers is studied for two influencing factors, aspect ratio of FE model and propagation angle. The chapter also provides a comprehensive guideline to prevent the appearance of fictitious modes within frequency range of interest.

Fig. 3.1: (A) Unit cell of M-layered composite in 2-D sagittal plane where wave can propagate in any direction by making angle \( \theta \) with the \( x_3 \)-axis. Periodic length of the inclined wave is given by \( a_3 \cos \theta \). (B-1) Reciprocal space representing wave motion perpendicular to the layer, where the top rectangle shows the first BZ. \( \Gamma - X \) denotes the actual wavevector and the two aliasing wavevectors are represented by \( \Gamma' - X' \) and \( \Gamma'' - X'' \). (B-2) Reciprocal space showing front and back aliasing zones where the inclined wavevector \( \Gamma - \Lambda \) produces aliasing wavevectors \( \Gamma_F - \Lambda_F \) and \( \Gamma_B - \Lambda_B \), respectively.

### 3.1 Spectral Distortion of Numerical Dispersion Relation

This section explains the spectral distortion in numerical dispersion relation. The nature of fictitious modes is explained at first which is followed by dispersion analysis procedure in FE framework.
3.1.1 Characteristics of Fictitious Modes in Dispersion Relation

This section first describes the effect of fictitious modes for wave propagation perpendicular to the multilayered composites. Then, the influence of oblique wave propagation on the fictitious modes is investigated.

3.1.1.1 Wave Motion Perpendicular to IPMC Layers

The FE method for dispersion analysis has been predominantly used for wave propagation perpendicular to the layers [2, 70, 117, 127]. Despite the simple geometry of layered composites (shown in Fig. 3.1A), the results of some numerical techniques for phononic dispersion relations for wave propagation perpendicular to the layers should be carefully interpreted. For infinitely periodic composite, Fig. 3.2B illustrates numerical dispersion relations calculated in the FE framework, and it shows some wave modes (marked by solid lines) which are not expected from the corresponding analytical solution in Fig. 3.2A. The appearance of those fictitious modes are known to be highly affected by the aspect ratio of the selected unit cell for the considered layered composites [2, 116, 117]. Since the composite in Fig. 3.1A is only periodic along $x_3$-axis, the periodic unit cell length $a_3$ is fixed in both numerical model. However, due to lack of periodicity in $x_1$-direction, the length $a_1$ is selected arbitrarily. For the unit cell having the aspect ratio of $a_1/a_3 = 2.0$, Fig. 3.2B-1 presents the results of the numerical dispersion analysis performed in the FE framework, where the blue lines with dot marker are the desired dispersion relations for waves perpendicular to the layers and the solid red lines are the fictitious modes. The fictitious modes are identified after comparing the FE solutions with analytical dispersion relations of Fig. 3.2A. On the other hand, the numerical results of dispersion analysis with the unit cells of $a_1/a_3 = 1.0$ shown in Fig. 3.2B-2 contains similar fictitious modes. However, the appearance of these fictitious modes differ from the model with aspect ratio $a_1/a_3 = 2.0$ (see Fig. 3.2B-1) Notice that the fictitious modes start to appear at higher frequencies as the aspect ratio $a_1/a_3$ decreases. This is why many researchers have been employing a thin unit cell [2, 116, 117] during the dispersion analysis.
3. Spatial Aliasing Solution of Numerical Dispersion Relation

![Fig. 3.2: (A) Analytical dispersion relation for wave propagation perpendicular to 3-layered composite. Effect of aspect ratio on the numerical dispersion relation in FE framework where model aspect ratios (B-1) $a_1/a_3 = 2.0$ and (B-2) $a_1/a_3 = 1.0$ are used.]

3.1.1.2 Wave Propagation Oblique to IPMC Layers

Due to the ability to develop intricate models, FE method is also suitable for performing numerical dispersion analysis of oblique wave motion in sagittal plane of IPMC. In order to obtain the dispersion relation for oblique wave propagation in 1-D periodic composite using FE platform, a 2-D FE model is compulsory to accommodate the direction of the wave motion in sagittal plane $x_1 - x_3$. For wave propagation oblique to the layers, the source of spectral distortion in dispersion relation obtained by FE method become more complex. This issue can be explained using the dispersion relations of Fig. 3.3. The analytical and numerical dispersion relations for wave motion angles $\theta = 0^\circ$, $45^\circ$ and $75^\circ$ are shown in Figs. 3.3A and 3.3B, respectively.

The numerical dispersion analysis for all three directions are performed for same aspect ratio of the FE model. In Fig. 3.3B for the numerical dispersion relations, the actual and fictitious dispersion relations represented by blue lines dot marker and red solid lines, respectively. Notice that the behavior of the fictitious modes for oblique wave motion differ from wave propagation perpendicular to the layers. For instance, the fictitious modes in the case of $\theta = 0^\circ$ originate at $\kappa_\theta = 0$ and reaches the Brillouin zone boundary at $\kappa_\theta = \pi/a_\theta$. But for wave motion angles $\theta = 45^\circ$ and $75^\circ$ the fictitious modes emanate as pairs from the origin wavenumber $\kappa_\theta = 0$ in Figs. 3.3B-2 and 3.3B-3, respectively. The origin of this feature of spatial aliasing in the dispersion relation also requires further explanation.
3. Spatial Aliasing Solution of Numerical Dispersion Relation

Fig. 3.3: Analytical dispersion relations of 3-layered composite for wave propagation angles (A-1) $\theta = 0^\circ$, (A-2) $\theta = 45^\circ$ and (A-3) $\theta = 75^\circ$. Effect of oblique wave motion on spectral distortion in numerical dispersion relations where same unit cell aspect ratio $a_1/a_3 = 0.4$ is used for wave propagation angles (A-1) $\theta = 0^\circ$, (A-2) $\theta = 45^\circ$ and (A-3) $\theta = 75^\circ$.

3.1.2 Dispersion Analysis in the FE Framework

The dispersion analysis procedure and modeling of unit cell is explained in FE framework. A comprehensive method for obtaining dispersion relation in sagittal plane of IPMC is presented in this section.

Although the dispersion relation of periodic structures in the FE platform [117, 127, 162] were investigated previously, the procedure to perform the analysis using the conventional FE program was developed [2] later. In this section we briefly review the method to obtain dispersion relation of phononic crystal using commercial FE programs[2, 171].

The solution of the dispersion relation depends on a complex eigenvalue problem. Since many FE programs do not support such complex-valued formulation, the displacement and stress fields of the periodic structure is decomposed into real and imaginary parts. Two identical FE structures are employed to represent the real and imaginary components of the Bloch periodic condition (2.9). Application of the periodic boundary condition only on the displacement field is sufficient in the FE framework, because it automatically imposes traction boundary condition.
By considering $\mathbf{R}$ as the vector connecting two periodic nodes located at the boundary of the unit cell, the decoupled Bloch periodic conditions which relate the real and imaginary structures is expressed in the form:

$$\begin{align*}
\mathbf{u}^{re}(\mathbf{x} + \mathbf{R}) &= \mathbf{u}^{re}(\mathbf{x}) \cos(\mathbf{\kappa} \cdot \mathbf{R}) - \mathbf{u}^{im}(\mathbf{x}) \sin(\mathbf{\kappa} \cdot \mathbf{R}) \\
\mathbf{u}^{im}(\mathbf{x} + \mathbf{R}) &= \mathbf{u}^{re}(\mathbf{x}) \sin(\mathbf{\kappa} \cdot \mathbf{R}) + \mathbf{u}^{im}(\mathbf{x}) \cos(\mathbf{\kappa} \cdot \mathbf{R})
\end{align*}$$

(3.1)

where superscripts $re$ and $im$ denote the real and the imaginary parts of the displacement field, respectively. Now, the dispersion relation of a periodic structure can be obtained from eigen-frequency analysis for entire range of the required wavevector $\mathbf{\kappa}$. In the present study, the eigen-frequency analysis is performed in ABAQUS/Standard where the Bloch periodic condition (3.1) is applied through a user defined subroutine MPC [1].

It is already mentioned that researchers use [2, 70, 73, 74, 117, 127] 2-D rectangular unit cell to determine the dispersion relation of periodic layered composites in FE framework. Since all the previous studies considered wave propagation perpendicular to the layers, the actual wavevector $\mathbf{\kappa}_\theta$ did not have any component along $\kappa_1$-axis. However, this is not the case for an obliquely propagating wave in periodic multilayered composite. Consider the multilayered periodic composite shown in Fig. 3.1A has a rectangular FE unit cell $a_1 \times a_3$ where $a_1$ is the length of the cell along $x_1$-axis. The dispersion relation for wave in sagittal plane $x_1 - x_3$ can be described in the reciprocal space by $\kappa_1 - \kappa_3$ plane. The irreducible Brillouin zone (IBZ) [105] for the 2-D FE model can be determined by considering the symmetry of the layered composite, and it results in wavevector domain of $\kappa_1 \in [0, \pi/a_1]$ and $\kappa_3 \in [0, \pi/a_3]$ (see Fig. 3.1B). Fig. 3.1B-1 shows the Brillouin zone where the wavevector $\mathbf{\kappa}_\theta$ denoted by $\Gamma - X$ propagates perpendicular to the layers. On the other hand, the wavevector $\mathbf{\kappa}_\theta$ in Fig. 3.1B-2 is inclined at angle $\theta$ with the $\kappa_3$-axis. According to (2.6), the reciprocal lattice vectors of first Brillouin zone in the $\kappa_3$ and $\kappa_1$ directions are shown in Fig. 3.1B-2 by $\Gamma - X$ and $\Gamma - Z_F$, respectively. The wavevector $\mathbf{\kappa}_\theta$ which is represented by $\Gamma - \Lambda$ in Fig. 3.1B-2, can be decomposed into $\Gamma - X$ and $X - \Lambda$ along $\kappa_3$ and $\kappa_1$-axes, respectively. While, the length of $\Gamma - X$ is fixed at $\pi/a_3$ by the periodic length $a_3$, the length $\kappa_1'$ (i.e., $\pi/a_3 \tan \theta$) of $X - \Lambda$ varies depending on angle $\theta$. Moreover, the value of $\kappa_1'$ is limited by $\pi/a_1$ which is the length of primitive lattice vector $X - L_B$. When $\kappa_1'$ attains...
maximum length $\pi/a_1$, the corresponding wave propagation angle is defined as the critical angle:

$$\theta_{\text{crit}} = \tan^{-1}\left(\frac{\pi/a_1}{\pi/a_3}\right) = \tan^{-1}\left(\frac{a_3}{a_1}\right),$$  (3.2)

which is a function of the unit cell aspect ratio. Consider that the unit cell in Fig. 3.1A has aspect ratio $a_1/a_3 = 0.4$ and therefore, the value of corresponding $\theta_{\text{crit}}$ is $68.2^\circ$. The critical angle $\theta_{\text{crit}}$ establishes the condition for two different dispersion analysis approaches of periodic layered composites in the FE framework. In the first case, the angle of wave motion is smaller than $\theta_{\text{crit}}$ and the dispersion relation is directly determined by solving the $\kappa_3$ and $\kappa_1$ components along $\Gamma - \Lambda$. However, in the event when wave propagation angle is greater than $\theta_{\text{crit}}$, the wavevector $\kappa_\theta$ (i.e., $\Gamma - \Lambda$) is dissociated into two parts by the BZ boundary. For instance, the wave in Fig. 3.4A-1 propagates at $75^\circ$ angle which is greater than $\theta_{\text{crit}}$ and intersects the first BZ at $Y$. The actual wavevector $\Gamma - \Lambda$ is separated into wavevectors $\Gamma - Y$ and $Y - \Lambda$, where only $\Gamma - Y$ is contained within first BZ. The wavevector segment $Y - \Lambda$ which falls outside the first BZ, should be traced back inside by the primitive reciprocal lattice vector $b_1$ (i.e., $2\pi/a_1$) to $Y' - \Lambda'$ as shown in Fig. 3.4A-1. So, the wavevector $\Gamma - \Lambda$ can be alternatively represented by the combination of $\Gamma - Y$ and $Y' - \Lambda'$.

Eventually, for the wavevectors $\Gamma - Y$ and $Y' - \Lambda'$ residing inside first BZ (see Figs. 3.4B-1 and 3.4C-1, respectively), one can separately find the corresponding dispersion relations shown in Figs. 3.4B-2 and 3.4C-2, respectively. The final dispersion relation of $\Gamma - \Lambda'$ shown in Fig. 3.4A-2 is obtained by merging the dispersion relations of $\Gamma - Y$ and $Y' - \Lambda'$ at $YY'$.

### 3.2 Spatial Sampling of Periodic Structure

It is widely known that sampling a continuous signal at discrete but equally spaced instants of time causes fictitious frequencies in the corresponding spectral domain due to the act of discretization. This spectral distortion is called temporal aliasing, and it occurs when a discrete Fourier series is employed for approximating a continuous periodic function [135]. Temporal aliasing is a well-known issue in signal processing, and it can be avoided if the sampling frequency is at least twice as large as the highest frequency contained within the continuous signal (or the highest frequency of interest). This principle is called the Nyquist theorem [135], and it
determines a proper sampling time interval (\(\Delta t\)) for the highest frequency of interest (\(\omega_{\text{max}}\)):

\[
\Delta t < \frac{\pi}{\omega_{\text{max}}}
\]  

(3.3)

The appearance of fictitious modes observed in the numerical dispersion relations of layered composites (i.e., 1-D phononic crystals) can be explained using the concept of spatial aliasing, which is analogous to temporal aliasing. In the FE framework, the phononic dispersion analysis requires the use of a rectangular unit cell in a 2-D coordinate space. Although the properties of layered composites are continuous in the direction parallel to the layers (i.e., the \(x_1\)-axis in Fig. 3.1A), the 2-D modeling for a unit cell in the FE procedure introduces spatial discretization in the \(x_1\)-axis, eventually causing fictitious modes in the corresponding wavevector domain (i.e., along the \(\kappa_1\)-axis). This study will demonstrate that the fictitious modes observed in numerical dispersion relations are the results of spatial aliasing in the wavevector domain. In this section, the issue of spatial discretization is presented to explain the spectral distortion in FE dispersion relations of infinitely periodic multilayered composites.
3.2.1 Discrete Fourier Transform

The spatial sampling of a function can be explained using discrete Fourier transform. The layered composite is envisioned as a 2-D crystal to obtain the dispersion relation in FE platform, where a rectangular unit cell $a_1 \times a_3$ (see Fig. 3.1A) is selected for the analysis. The crystal can be considered as a spatial function $\chi(x)$ which contains $N_1$ and $N_3$ unit cells in $x_1$ and $x_3$ directions, respectively. Hence, a total of $N_1 \times N_3$ unit cells are available in the crystal which are sampled at $\Delta x_1 = |a_1| = a_1$ and $\Delta x_3 = |a_3| = a_3$ along $x_1$ and $x_3$-axes, respectively. Now, the function $\chi(x)$ can be expressed as discrete series $\{\chi[r_1, r_3]\}$ (for $r_k = 1, 2, \ldots, N_k - 1$ where $k = 1, 3$) in the spatially discretized region $x_1 \in [0, (N_1 - 1)\Delta x_1]$ and $x_3 \in [0, (N_3 - 1)\Delta x_3]$. The wavevector domain components for this discrete spatial function is obtained by discrete Fourier transform (DFT) [29] as:

$$\hat{\chi}[g_1, g_3] = \frac{1}{N_1N_3} \sum_{r_1=0}^{N_1-1} \sum_{r_3=0}^{N_3-1} \chi[r_1, r_3] e^{-i2\pi \left(\frac{r_1g_1}{N_1} + \frac{r_3g_3}{N_3}\right)}$$

(3.4)

where $g_k = 0, 1, \ldots, N_k - 1$ for $k = 1, 3$ and $\hat{\chi}$ specifies wavevector domain components. The spatial sampling of $\{\chi[r_1, r_3]\}$ by $\Delta x_1$ and $\Delta x_3$ induces reciprocal space periodicity at $\Delta \kappa_1 = 2\pi/\Delta x_1$ and $\Delta \kappa_3 = 2\pi/\Delta x_3$ along $\kappa_1$ and $\kappa_3$ wavevector axes, respectively. Therefore the coefficients of $\hat{\chi}[g_1, g_3]$ has the periodic property:

$$\hat{\chi}[g_1 + l_1\Delta \kappa_1, g_3 + l_3\Delta \kappa_3] = \hat{\chi}[g_1, g_3]$$

(3.5)

where $l_1$ and $l_3$ are arbitrary integers. Additionally, the complex Fourier transform operation on the real function $\chi(x)$ produces conjugate symmetry about the center of smallest reciprocal unit $(\Delta \kappa_3/2, \Delta \kappa_1/2)$ as:

$$\hat{\chi}[N_3 - g_3, N_1 - g_1] = \hat{\chi}^*[g_3, g_1]$$

(3.6)

where $\Box^*$ indicates the complex conjugate. Therefore, the magnitude of the DFT coefficients $|\hat{\chi}[g_1, g_3]|$ is now symmetric around multiples of $\pi/\Delta x_1$ and $\pi/\Delta x_3$ in the $\kappa_1$- and $\kappa_3$-axes, respectively. This symmetry is commonly referred as the zone folding [33], and the folding wavenumbers $\kappa_{1,f}$ and $\kappa_{3,f}$ are defined as:

$$\kappa_{1,f} = \frac{\pi}{\Delta x_1}, \quad \kappa_{3,f} = \frac{\pi}{\Delta x_3}.$$  (3.7)
The entire property of the crystal can be obtained by analyzing the reciprocal space enclosed by \( \kappa_3 \in [0, \kappa_{3,f}] \) and \( \kappa_1 \in [0, \kappa_{1,f}] \). Only the wavevector coefficients \( \hat{\chi}[g_3, g_1] \) residing inside the first BZ are unique. The remaining coefficients that can be found from (3.5) corresponds to aliasing zones of first BZ. This periodicity of the DFT entails infinitely many aliases of the true spectrum in the wavevector domain.

### 3.2.2 Spatial Aliasing

Assume that the spatial function \( \chi(x) \) in Section 3.2.1 contains wavenumber components beyond the wavevector domain of \( \kappa_1 \in [0, \pi/\Delta x_1] \) and \( \kappa_3 \in [0, \pi/\Delta x_3] \). Then, these large wavenumber components falsely contribute to the DFT coefficient series calculated from the discretely measured data \( \{\chi[r_1, r_3]\} \). This numerical issue is referred as spatial aliasing [29], and it can be alleviated by increasing the discrete sampling rate which enables to capture high-wavenumber components existing in the continuous function.

In order to calculate the phononic dispersion relation of an infinitely periodic multilayered composite in the FE framework, a rectangular unit cell of \( a_1 \times a_3 \) has to be employed (see Fig. 3.1A) with periodic boundary condition along \( x_1 \) and \( x_3 \)-axes. Note that the properties of the layered composite are \( a_3 \)-periodic in the \( x_3 \)-axis (i.e., the direction perpendicular to the layers shown in Fig. 3.1A) while they are continuous along the \( x_1 \)-axis (i.e., the direction parallel to the layers shown in Fig. 3.1A). The wave characteristics of the multilayered composite relating to \( a_3 \)-periodicity can be properly captured by using a unit cell length of \( \Delta x_3 = a_3 \) in the direction perpendicular to the layers. However, \( \kappa_1 \) components relating to homogeneous properties in \( x_1 \)-direction cannot be fully accessed by using a unit cell length of \( \Delta x_1 = a_1 \) in the direction parallel to the layers. Specifically, this coarse spatial discretization in the \( x_1 \)-axis induces the artificial \( 2\pi/a_1 \)-periodicity along the \( \kappa_1 \)-axis of the wavevector domain and the calculated spectral behavior is distorted by the existence of the high-wavenumber components from a infinite number of aliasing zones of the first Brillouin zone (see Fig. 3.1B).

For instance, the wavevector path denoted by \( \Gamma - X \) for wave motion perpendicular to the layers, Fig. 3.1B-1 shows an infinite number of aliasing paths, i.e., \( \Gamma' - X', \Gamma'' - X'', \Gamma''' - X''' \), etc. On the other hand, for wave motion oblique to the periodic layers, Fig. 3.1B-2 shows that the dispersion relation is affected by different fictitious modes from identical front and back aliasing zones. This incident is illustrated using reciprocal space in Fig. 3.11B, which shows that the actual wavevector \( \Gamma - \Lambda \) has two identical front and back aliasing wavevectors \( \Gamma'_F - \Lambda'_F \) and...
3. Spatial Aliasing Solution of Numerical Dispersion Relation

\[ \Gamma_B' - \Lambda_B' \], respectively. This study first focuses on the aspect ratio of the model which affect the folding wavenumber \( \kappa_{1,f} = \pi \ a_1 \) of Fig. 3.1B-1. Then, by fixing the model aspect ratio, the effect of change of angle \( \theta \) is studied.

3.3 Dispersion Analysis Results

This section describes the details of the materials and geometries considered for investigating the spatial aliasing in infinitely periodic multilayered composites. Both the influencing factors, aspect ratio and propagation angle are studied and the spatial aliasing effect is explained for corresponding cases.

3.3.1 Material and Geometry

Two different infinitely periodic multilayered composites are selected to investigate the origin of the spectral distortions observed in FE dispersion relations: one with an infinitely periodic 3-layered composite and the other with an infinitely periodic 4-layered one. Basically, these two composite cases are provided to show that the spatial aliasing solution presented in this study is applicable to any infinitely periodic multilayered composites regardless of the number of periodic layers. After explaining the effect of aspect ratio with two composites, the spatial aliasing behavior of oblique wave propagation is presented for the 3-layered composite. The unit cell of the infinitely periodic 3-layered composite consists of steel \((d_1 = 0.4 \ mm)\), aluminum \((d_2 = 0.4 \ mm)\) and copper \((d_3 = 0.2 \ mm)\), so the periodic length of the composite in the \(x_3\)-axis is \(a_3 = \sum_{j=1}^{3} d_j = 1 \ mm\). On the other hand, for the infinitely periodic 4-layered composite, the unit cell is composed of steel \((d_1 = 0.4 \ mm)\), aluminum \((d_2 = 0.2 \ mm)\), copper \((d_3 = 0.2 \ mm)\) and titanium \((d_4 = 0.2 \ mm)\), resulting in the periodic length of \(a_3 = 1 \ mm\). The properties of the considered materials are listed in Table 3.1.

As discussed in Sec. 3.1.2, the numerical procedure for dispersion analysis in the FE framework requires the use of a rectangular unit cell in the 2-D coordinate space. In order to highlight the effect of the aspect ratio of \(a_1/a_3\) in the considered unit cell, two different aspect ratios \(a_1/a_3 = 2.0\), and \(a_1/a_3 = 1.0\) are used for the both composites. Figures 3.5A and 3.6A describe the geometries and the corresponding Brillouin/aliasing zones of the infinitely periodic 3-layered composite models with \(a_1/a_3 = 2.0\) and \(a_1/a_3 = 1.0\), respectively. The analogous schematic diagrams for the infinitely periodic 4-layered models with \(a_1/a_3 = 2.0\) and \(a_1/a_3 = 1.0\) are also
shown in 3.7A and 3.8A, respectively.

Plane strain conditions are assumed in all the FE simulations, so all the 2-D FE models are constructed using 4-node bilinear plane strain elements (CPE4R elements in ABAQUS). The detailed procedure to obtain dispersion relation using the FE method can be found in [73, 171]. Through a mesh refinement study, we select the mesh size of 0.025a3 to ensure the convergence of the FE simulations.

<table>
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<th>Index, j</th>
<th>Material</th>
<th>( \rho ) [kg/m(^3)]</th>
<th>( \lambda ) [GPa]</th>
<th>( \mu ) [GPa]</th>
</tr>
</thead>
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<td>121.2</td>
<td>80.8</td>
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<td>Aluminum</td>
<td>2700</td>
<td>51.1</td>
<td>26.3</td>
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<td>47.7</td>
</tr>
<tr>
<td>4</td>
<td>Titanium</td>
<td>4500</td>
<td>78.1</td>
<td>43.9</td>
</tr>
</tbody>
</table>

### 3.3.2 Effect of Aspect Ratio on Numerical Dispersion Relation

The numerical dispersion relations are presented for different aspect ratios of two composites and the spatial aliasing is explained using analytical dispersion relation of sagittal plane wave.

#### 3.3.2.1 Numerical Dispersion Relation Results

For the infinitely periodic 3-layered composite, Figs. 3.5B and 3.6B illustrate the numerical dispersion relations obtained using the FE models with the aspect ratios of \( a_1/a_3 = 2.0 \) and \( a_1/a_3 = 1.0 \), respectively. Notice that these two figures show two types of lines. The first type is the set of (blue) solid lines with dot markers which represents the correct dispersion relation for waves propagation perpendicular to the layers. The second set of the (red) solid lines represent fictitious modes. In the FE framework, those two types of modes are obtained simultaneously and they cannot be separated. These two types of modes are separated by comparing FE results with the analytical solution, which will be discussed in Sec. 3.3.2.2. From the FE model with the aspect ratio of \( a_1/a_3 = 2.0 \), the first fictitious mode appears around the normalized frequency of \( \tilde{\omega} = \omega a_3/(2\pi c_{s,2}) = 0.5 \). On the other hand, the FE model with \( a_1/a_3 = 1.0 \) produces the first fictitious modes around \( \tilde{\omega} = 1.0 \). As the aspect ratio of the unit cell (i.e., \( a_1/a_3 \)) increases, the first fictitious mode tends to emerge at lower frequencies. For the infinitely periodic 4-layered composite, a similar trend is also observed in the numerical dispersion relations for
wave propagation perpendicular to the layers. In Figs. 3.7B and 3.8B, the first fictitious mode is observed around $\bar{\omega} = 0.5$ and $\bar{\omega} = 1.0$ from the FE model with $a_1/a_3 = 2.0$ and $a_1/a_3 = 1.0$, respectively. Note that any aspect ratio (even smaller than 1.0) can also be explored, but the FE dispersion relations will contain fictitious modes regardless of the magnitude of the aspect ratio (e.g., $[73]$). The two aspect ratios (i.e., $a_1/a_3 = 2.0, 1.0$) in this study are merely chosen to efficiently demonstrate the effect of the aspect ratio on the appearance of fictitious modes.

### 3.3.2.2 Spatial Aliasing Explanation

Using the analytical solution of sagittal plane waves (2.42), the dispersion relations of infinitely periodic multilayered composites can be calculated along specific wavevector paths. Due to the homogeneous properties along $x_1$-direction, wave motion in the infinitely periodic multilayered composites contains infinitely large $\kappa_1$ components. However, the spatial discretization by using a rectangular unit cell of $a_1 \times a_3$ in the FE framework induces the artificial $2\pi/a_1$-periodicity in the $\kappa_1$-axis of the wavevector domain as discussed in Sec. 3.2. Furthermore, it entails an infinitely many aliases of the true phononic dispersion relation, causing spectral distortions in numerical dispersion relation. In the following, the elastic version of sagittal plane wave solution (2.42) is adopted to reproduce the distorted dispersion relations shown in Section 3.3.2.

The infinitely periodic 3-layered composites are firstly considered. Recall that Fig. 3.5B shows the FE dispersion relation obtained by using a unit cell having $a_1/a_3 = 2.0$. Using the sagittal wave solution (2.42), we obtain the analytical dispersion relations at three different $\kappa_1$ values, all of which are multiples of $2\kappa_{1,f} = 2\pi/a_1$. In other words, analytical dispersion relation with $\kappa_1 = 0$ (i.e., the path $\Gamma - X$ in Fig. 3.5-A) is shown in Fig. 3.5C-1, and it represents the dispersion relation for waves perpendicular to the layers. In addition, Figs. 3.5C-2 and 3.5C-3 illustrate the sagittal wave solution (2.42) at $\kappa_1 = 2\kappa_{1,f}$ (i.e., $\Gamma' - X'$) and $\kappa_1 = 4\kappa_{1,f}$ (i.e., $\Gamma'' - X''$), respectively. Then, all the analytical dispersion relations calculated at different $\kappa_1$ values ($\kappa_1 = 0, 2\kappa_{1,f}, 4\kappa_{1,f}, 6\kappa_{1,f},$ etc.) are projected onto the $\kappa_3 - \omega$ plane, and the projected dispersion relation is shown in Fig. 3.5D. By comparing the Figs. 3.5B and 3.5D, one can find one-to-one correspondence in all the observed wave modes. Thus, the comparison between Figs. 3.5B and 3.5D shows that spectral distortions in the FE dispersion relations are induced by spacial aliasing originated from $\Gamma' - X'$, $\Gamma'' - X''$, etc., which are the aliasing paths of $\Gamma - X$ in the first Brillouin zone. Similarly, in Fig. 3.6, the numerical dispersion relation obtained from the FE model with a unit cell having the aspect ratio of $a_1/a_3 = 1.0$ (i.e., Fig. 3.6B) are
Fig. 3.5: (A) [Left] A unit cell of the infinitely periodic 3-layered composite having $a_1/a_3 = 2.0$, which is employed for numerical dispersion analysis. [Right] The corresponding wavevector domain, which illustrates the valid wavevector path for waves perpendicular to the layers ($\Gamma - X$) and the corresponding aliasing paths ($\Gamma' - X'$, $\Gamma'' - X''$). (B) FE dispersion relation obtained by employing a unit cell of $a_1/a_3 = 2.0$. Note that the numerical dispersion relation contains unwanted fictitious modes represented by red solid lines. (C) Three analytical dispersion relations obtained from (2.42): (C-1) $\kappa_1 = 0$, (C-2) $\kappa_1 = 2\pi/a_1$, and (C-3) $\kappa_1 = 4\pi/a_1$. (D) The projection of all the analytical dispersion relations in Fig. 3.5C onto the $\kappa_3 - \omega$ plane. Note that one can find one-to-one map in all the observed wave modes in Figs. 3.5B and 3.5D, indicating that the fictitious modes originate from the aliasing paths. Moreover, the maximum valid frequency $\omega_{max}$ from (3.16) is denoted by a line with square markers.
Fig. 3.6: (A) [Left] A unit cell of the infinitely periodic 3-layered composite having $a_1/a_3 = 1.0$, which is employed for numerical dispersion analysis. [Right] The corresponding wavevector domain, which illustrates the valid wavevector path for waves perpendicular to the layers $(\Gamma - X)$ and the corresponding aliasing paths $(\Gamma' - X', \Gamma'' - X'')$. (B) FE dispersion relation obtained by employing a unit cell of $a_1/a_3 = 1.0$. Note that the numerical dispersion relation contains unwanted fictitious modes represented by red solid lines. (C) Three analytical dispersion relations obtained from (2.42): (C-1) $\kappa_1 = 0$, (C-2) $\kappa_1 = 2\pi/a_1$, and (C-3) $\kappa_1 = 4\pi/a_1$. (D) The projection of all the analytical dispersion relations in Fig. 3.6C onto the $\kappa_3 - \omega$ plane. Note that one can find one-to-one map in all the observed wave modes in Figs. 3.6B and 3.6D, indicating that the fictitious modes originate from the aliasing paths. Moreover, the maximum valid frequency $\omega_{max}$ from (3.16) is denoted by a line with square markers.
3. Spatial Aliasing Solution of Numerical Dispersion Relation

Fig. 3.7: (A) [Left] A unit cell of the infinitely periodic 4-layered composite having $a_1/a_3 = 2.0$, which is employed for numerical dispersion analysis. [Right] The corresponding wavevector domain, which illustrates the valid wavevector path for waves perpendicular to the layers ($\Gamma - X$) and the corresponding aliasing paths ($\Gamma' - X'$, $\Gamma'' - X''$). (B) FE dispersion relation obtained by employing a unit cell of $a_1/a_3 = 2.0$. Note that the numerical dispersion relation contains unwanted fictitious modes represented by red solid lines. (C) Three analytical dispersion relations obtained from (2.42): (C-1) $\kappa_1 = 0$, (C-2) $\kappa_1 = 2\pi/a_1$, and (C-3) $\kappa_1 = 4\pi/a_1$. (D) The projection of all the analytical dispersion relations in Fig. 3.7C onto the $\kappa_3 - \omega$ plane. Note that one can find one-to-one map in all the observed wave modes in Figs. 3.7B and 3.7D, indicating that the fictitious modes originate from the aliasing paths. Moreover, the maximum valid frequency $\omega_{max}$ from (3.16) is denoted by a line with square markers.
3. Spatial Aliasing Solution of Numerical Dispersion Relation

Fig. 3.8: (A) [Left] A unit cell of the infinitely periodic 4-layered composite having \( a_1/a_3 = 1.0 \), which is employed for numerical dispersion analysis. [Right] The corresponding wavevector domain, which illustrates the valid wavevector path for waves perpendicular to the layers (\( \Gamma - X \)) and the corresponding aliasing paths (\( \Gamma' - X', \Gamma'' - X'' \)). (B) FE dispersion relation obtained by employing a unit cell of \( a_1/a_3 = 1.0 \). Note that the numerical dispersion relation contains unwanted fictitious modes represented by red solid lines. (C) Three analytical dispersion relation obtained from (2.42): (C-1) \( \kappa_1 = 0 \), (C-2) \( \kappa_1 = 2\pi/a_1 \), and (C-3) \( \kappa_1 = 4\pi/a_1 \). (D) The projection of all the analytical dispersion relations in Fig. 3.8C onto the \( \kappa_3 - \omega \) plane. Note that one can find one-to-one map in all the observed modes in Figs. 3.8B and 3.8D, indicating that the fictitious modes originate from the aliasing paths. Moreover, the maximum valid frequency \( \omega_{max} \) from (3.16) is denoted by a line with square markers.
3. Spatial Aliasing Solution of Numerical Dispersion Relation

compared with the analytical dispersion relations obtained from sagittal plane waves (i.e., Fig. 3.6D). Furthermore, for the infinitely periodic 4-layered composites, Figs. 3.7 and 3.8 compare the sagittal plane wave dispersion relations with the numerical result obtained from the FE models having the aspect ratios of $a_1/a_3 = 2.0$ and 1.0, respectively.

The results presented in Figs. 3.5 - 3.8 clearly show that a smaller aspect ratio in FE modeling introduces fictitious modes at higher frequencies, thus resulting in fewer fictitious modes in the presented graphs. The sagittal plane wave solutions prove that the fictitious modes occurs at the aliasing paths. In the wavenumber domain, the distance of the aliasing paths (i.e., $\Gamma' - X'$, $\Gamma'' - X''$, etc.) from the wavevector path $\Gamma - X$ within the first Brillouin zone can be expressed by:

$$\kappa_{1,l} = \frac{2\pi l}{a_1} \tag{3.8}$$

where $l$ is integer. Thus, as the aspect ratio decreases (i.e., $a_1$ decreases), $\kappa_{1,l}$ increases (see Fig. 3.2 B). Consequently, due to the proportional relation between wavenumber and the corresponding frequency in elastic wave motion, fictitious modes appears at higher frequencies as $a_1$ decreases.

3.3.3 Effect of Propagation Angle on Numerical Dispersion Relation

The aspect ratio is fixed to evaluate the effect of propagation direction on the numerical dispersion relations. Similar to the Sec. 3.3.2, the analytical dispersion relation of sagittal plane wave is employed to reproduce complete numerical dispersion relations.

3.3.3.1 Numerical Dispersion Relation Results

The attributes of numerical dispersion relation for oblique wave propagation are evaluated using two different propagation angles $45^\circ$ and $75^\circ$ by fixing aspect ratio of the periodic 3-layered composite at $a_1/a_3 = 0.4$ as shown in Figs. 3.9A and 3.10A, respectively. Figs. 3.9B and 3.10B show the dispersion relations for wave propagation angles $45^\circ$ and $75^\circ$ which lie above and below the critical angle $\theta_{\text{cr}}$, respectively. In these numerical dispersion relation results, the actual and fictitious dispersion modes are distinguished by dot marked lines and solid lines, respectively. Notice that the fictitious modes emanate as pairs whose origin frequencies are located at the
wavenumber $\kappa_\theta = 0$ in both dispersion relations in Figs. 3.9B and 3.10B. Interestingly, one fictitious mode approaches toward lower frequency range and another commences to upper frequency region which are referred as lower and upper fictitious modes, respectively. In fact, the origin frequencies (i.e., at $\kappa_\theta = 0$) of the fictitious mode pairs remain same for the selected aspect ratio $a_1/a_3 = 0.4$ and develop differently depending on the wave propagation angle. For instance, at $\kappa_\theta = 0$, the origin of first fictitious frequency is found around $\bar{\omega} = 2.0$, but the lower fictitious mode reaches frequency $\bar{\omega} = 1.7$ and $\bar{\omega} = 0.6$ at the BZ boundary (i.e., $\kappa = \pi/a_\theta$) in Figs. 3.9B and 3.10B, respectively.

Therefore, fictitious modes of the dispersion relation are not only a function of the aspect ratio of the FE model, but also depend on the inclination angle when wave propagates oblique to the periodic multilayered composite. Since all the FE based dispersion analyses are performed for wave propagation perpendicular to the layers [2, 73, 74, 116, 117], this aspect of the spectral distortion is completely unknown to the researchers. Consequently, the explanation of Sec. 3.3.2.2 on spectral distortion is unable to interpret the issue and further exploration is required to obtain the solution. Proper identification of the fictitious modes is necessary because it can easily influence the band-gap detection. Specially, when a fictitious mode span in a band-gap region, it either reduces the range of band-gap frequency or completely suppress it.

### 3.3.3.2 Spatial Aliasing Explanation

This section explains the fictitious modes in the numerical dispersion relation using the analytical solution (??) for oblique wave propagation in the sagittal plane of periodic multilayered composites. The source of aliasing zones in reciprocal space and corresponding aliasing wavevectors are introduced in Sec. 3.2.2. The notable aspect of the spectral distortion for oblique wave propagation is that the dispersion relation is affected by different fictitious modes from identical front and back aliasing zones as shown in Fig. 3.1C. This incident is illustrated by using reciprocal space in Fig. 3.11B, which shows that the actual wavevector $\Gamma - \Lambda$ has two identical front and back aliasing wavevectors $\Gamma'_F - \Lambda'_F$ and $\Gamma'_B - \Lambda'_B$, respectively. Although all the wavevectors have same direction of propagation, $\Gamma'_F - \Lambda'_F$ and $\Gamma'_B - \Lambda'_B$ have different wavenumber component $\kappa_1$ for identical points. Notice that both the aliasing wavevectors $\Gamma'_F - \Lambda'_F$ and $\Gamma'_B - \Lambda'_B$ initiate at the same distance $2\pi/a_1$ (i.e., $\Delta\kappa_1$) from $\kappa_3$-axis. As the wavevectors proceed in the reciprocal space, the net aliasing wavenumber decreases for $\Gamma'_F - \Lambda'_F$ and increases for $\Gamma'_B - \Lambda'_B$. Therefore, along the aliasing axis $\kappa_1$, the wavevectors $\Gamma'_F - \Lambda'_F$ and $\Gamma'_B - \Lambda'_B$ span differently over the ranges
Fig. 3.9: (A) [Left] A unit cell of the infinitely periodic 3-layered composite having $a_1/a_3 = 0.4$ where wave propagates at $\theta = 45^\circ$. [Right] The corresponding wavevector domain, which illustrates the valid wavevector path $\Gamma - \Lambda$ and the corresponding front and back aliasing paths $\Gamma_F' - \Lambda_F'$, $\Gamma_B' - \Lambda_B'$, respectively. (B) FE dispersion relations where the numerical dispersion relation contains unwanted fictitious modes represented by red solid lines. (C) Three analytical dispersion relation obtained from (2.42) for wavevectors (C-1) $\Gamma - \Lambda$, (C-2) $\Gamma_B' - \Lambda_B'$, and (C-3) $\Gamma_F' - \Lambda_F'$. (D) The projection of all the analytical dispersion relations in Fig. 3.9C onto the $\kappa_\theta - \omega$ plane. Note that one can find one-to-one map in all the observed modes in Figs. 3.9B and 3.9D, indicating that the fictitious modes originate from the aliasing paths. Moreover, the maximum valid frequency $\omega_{\text{max}}$ from (3.16) is denoted by a line with square markers.
Fig. 3.10: (A) [Left] A unit cell of the infinitely periodic 3-layered composite having $a_1/a_3 = 0.4$ where wave propagates at $\theta = 75^\circ$. [Right] The corresponding wavevector domain, which illustrates the valid wavevector path $\Gamma - \Lambda$ and the corresponding front and back aliasing paths $\Gamma'_F - \Lambda'_F$, $\Gamma'_B - \Lambda'_B$, respectively. (B) FE dispersion relations where the numerical dispersion relation contains unwanted fictitious modes represented by red solid lines. (C) Three analytical dispersion relation obtained from (2.42) for wavevectors (C-1) $\Gamma - \Lambda$, (C-2) $\Gamma'_B - \Lambda'_B$, and (C-3) $\Gamma'_F - \Lambda'_F$. (D) The projection of all the analytical dispersion relations in Fig. 3.10C onto the $\kappa_0 - \omega$ plane. Note that one can find one-to-one map in all the observed modes in Figs. 3.10B and 3.10D, indicating that the fictitious modes originate from the aliasing paths. Moreover, the maximum valid frequency $\omega_{max}$ from (3.16) is denoted by a line with square markers.
\( \kappa_1 \in \left[ \frac{2\pi}{a_1}, \frac{2\pi}{a_1} - \kappa'_1 \right] \) and \( \kappa_1 \in \left[ \frac{2\pi}{a_1}, \frac{2\pi}{a_1} + \kappa'_1 \right] \), receptively. Evidently, \( \Gamma'_F - \Lambda'_F \) and \( \Gamma'_B - \Lambda'_B \) have different set of wavenumbers except the origin \( \frac{2\pi}{a_1} \). The difference in wavevectors explains the pair of fictitious modes starting from a single origin frequency in the dispersion relation. In contrast, for wave propagation perpendicular to the layers, the wavevectors from identical front and back aliasing zones contain same set of wavenumbers and the contributing fictitious modes overlap each other.

In order to verify the proposed hypothesis about spatial aliasing in dispersion relation for oblique wave propagation, the analytical sagittal plane wave solution (2.42) is employed. The dispersion relation of the actual wavevector is calculated by using the wavevector condition \( \kappa_1 = \kappa_3 \tan \theta \). Moreover, the front and back aliasing wavevectors can be also found by analyzing the corresponding \( \kappa_1 \) ranges:

\[
\kappa_1 \in \left[ l_1 \Delta \kappa_1, l_1 \Delta \kappa_1 - \kappa'_1 \right], \quad \kappa_1 \in \left[ l_1 \Delta \kappa_1, l_1 \Delta \kappa_1 + \kappa'_1 \right]
\]

(3.9)

where \( \Delta \kappa_1 = \frac{2\pi}{a_1} \) and \( l_1 \) represents the sequence of aliasing zone (i.e., \( l_1 = 1, 2, ..., \infty \)). The results of the spatial aliasing can be explained using Figs. 3.9 and 3.10 for wave propagation at 45° and 75° angles, respectively. At first, consider the wave propagation at 45° angle (see Fig. 3.9A) in the 3-layered periodic composite with unit cell aspect ratio \( a_1/a_3 = 0.4 \). The analytical results are presented using a 3-D coordinate system of \( \kappa_3 - \kappa_1 - \omega \) in Fig. 3.9C, where the dispersion relations are plotted on a 45° inclined plane. The actual dispersion relation of \( \Gamma - \Lambda \) is illustrated in Fig. 3.9C-1. The dispersion relations of the aliasing wavevectors \( \Gamma'_F - \Lambda'_F \) and \( \Gamma'_B - \Lambda'_B \) are obtained using the \( \kappa_1 \) ranges given by (3.9) for \( l_1 = 1 \) which are shown in Figs. 3.9C-2 and 3.9C-3, respectively. Note that, Fig. 3.9B represents the dispersion relation of the same oblique wave in FE platform which contains both actual and fictitious modes. In fact, this numerical dispersion relation can be produced by projecting the analytical results of Fig. 3.9C in one plane to obtain the dispersion relation of Fig. 3.9D. In Fig. 3.9D, the actual dispersion modes of \( \Gamma - \Lambda \) are represented by blue lines and aliasing modes of \( \Gamma'_F - \Lambda'_F \) and \( \Gamma'_B - \Lambda'_B \) are shown by green and red lines, respectively. Notice that the fictitious modes of Fig. 3.9 correspond to the aliasing wavevector modes of Fig. 3.9D.

Similarly, Fig. 3.10 illustrates the dispersion relation results for wave propagation at 75° angle in the periodic multilayered composites. Since, the angle of wave motion 75° is greater than \( \theta_{\text{cric}} \), the numerical dispersion relation of Fig. 3.10B is obtained using the two step method
discussed in Sec. ?? From comparison of numerical dispersion relation presented in Fig. 3.10B with the projected analytical solution of Fig. 3.10D, it is found that the actual and fictitious modes are successfully identified. Note that this interpretation of spatial aliasing is performed for the frequency limit $\tilde{\omega} = 2.5$, where only the first spatial aliasing zones impair the dispersion relation. In fact, infinite number of aliasing zones contribute to the numerical dispersion relation which appears at higher frequency ranges. Indeed, the current explanation affirms that the fictitious modes in numerical dispersion relation are function of both aspect ratio and direction of wave motion. The direction of wave propagation is critical to evaluate the frequency band-gap of multilayered composites, because the band-gap property changes depending on the angle of wave motion.

In summary, aliasing-induced spectral distortions in FE dispersion relations are reproduced through the newly derived sagittal plane wave solution for infinitely periodic multilayered composites (2.42).

![Fig. 3.11: (A) Unit cell of M-layered composite in 2-D sagittal plane where $a_1$ and $a_3$ denotes the lattice vectors. The periodic length of the inclined wave is given by $a_3 \cos \theta$. (B) The front and back aliasing wavevectors $\Gamma_F' - \Lambda_F'$ and $\Gamma_B' - \Lambda_B'$ in reciprocal space which located at $2\pi/a_1$ from corresponding points of actual wavevector $\Gamma - \Lambda$ along $\kappa_1$-axis. The smallest aliasing wavenumber is represented by the path $X - \Lambda_F'$ at distance $2\pi/a_1 - \kappa_1'$ from $\kappa_3$-axis.](image)
3.4 Generalized Guideline for Spatial Aliasing

This section provides a generalized guideline by using anti-aliasing condition which identifies the proper unit cell configuration to obtain accurate numerical dispersion relation for a desired frequency range.

3.4.1 Anti-aliasing Condition

Sec. 3.2.2 reveals that the spatial discretization with $\Delta x_1 = a_1$ induces a symmetry along the folding wavenumber $\kappa_{1,f} = \pi / \Delta x_1 = \pi / a_1$. For the dispersion relation of infinitely periodic multilayered composites, spectral distortion induced by spatial aliasing occurs at an infinite number of the aliasing paths which are located at a multiple of $2\kappa_{1,f}$ in the $\kappa_1$-axis, i.e., $\kappa_1 = 2\kappa_{1,f}$, $4\kappa_{1,f}$, $6\kappa_{1,f}$, etc. If a unit cell of $a_1 \times a_3$ is employed for the FE modeling of the layered composite shown in Fig. 3.1B, the corresponding first fictitious mode for wave propagation perpendicular to the layers appears at:

$$\kappa_{1,max} < 2\kappa_{1,f} = \frac{2\pi}{a_1} \iff a_1 < \frac{2\pi}{\kappa_{1,max}}. \quad (3.10)$$

where $\kappa_{1,max}$ is the maximum wavenumber that the numerical dispersion relations can be valid up to. The second equation of (3.10) indicates a proper unit cell size $a_1$ in the FE framework for a given wavenumber of interest, $\kappa_{1,max}$. Recall that Sec. 3.3 explained the requirement of spatial attention to the spectral distortion for oblique wave propagation, a guideline based only on the relation (3.10) is deemed unsuitable. In fact, the valid frequency range of dispersion relation is limited by the appearance of first fictitious mode which corresponds to the minimum aliasing wavenumber. Consequently, the anti-aliasing condition of (3.10) represents the extent of maximum accurate wavenumber limit $\kappa_{1,max}$ for $\theta = 0^\circ$.

However, in case of oblique wave motion, this wavenumber belongs to one of the aliasing wavevectors $\Gamma_F - \Lambda'_F$ and $\Gamma'_B - \Lambda'_B$ from first front and back aliasing zones, respectively. According to Sec. 3.3.3.2, the wavenumber components along $\kappa_1$-axis for $\Gamma'_F - \Lambda'_F$ and $\Gamma'_B - \Lambda'_B$ have the ranges:

$$\kappa_1 \in [\Delta \kappa_1, \Delta \kappa_1 - \kappa'_1] \quad and \quad \kappa_1 \in [\Delta \kappa_1, \Delta \kappa_1 + \kappa'_1], \quad (3.11)$$

respectively. For any given aspect ratio $a_1/a_3$ and wave propagation angle $\theta$, (3.11) suggests that
the target aliasing wavenumber limit is:

\[ \kappa_{1,max} < \Delta \kappa_1 - \kappa'_1 = \frac{2\pi}{a_1} - \frac{\pi}{a_3} \tan \theta \quad . \tag{3.12} \]

Furthermore, Fig. 3.11B verifies that the length of \(X - \Lambda'_F\) represents the smallest distance of either \(\Gamma'_F - \Lambda'_F\) and \(\Gamma'_B - \Lambda'_B\) from \(\kappa_3\)-axis. Certainly, the anti-aliasing condition (3.12) indicates the wavenumber limit up to which the dispersion relation is devoid of fictitious modes for wave motion at angle \(\theta\) in the multilayered unit cell \(a_1 \times a_3\). In other words, as long as the unit cell configuration meets the criterion (3.12), the employed FE model is adequate to capture the spectral behavior of the layered composites subjected to a harmonic excitation with a long wavelength, which is greater than \(\lambda_{1,min}\) (i.e., \(2\pi/\kappa_{1,max}\)).

### 3.4.2 Effective Modulus Theory

Although the condition (3.12) advises the allowable wavenumber \(\kappa_{1,max}\), it does not present the accurate frequency limit of the dispersion relation. Generally, the most common practice of representing the dispersion analysis for multilayered periodic composites is by the \(\omega - \kappa_\theta\) relation. In order to obtain a complete guideline, \(\kappa_{1,max}\) from (3.12) should be related to the maximum accurate frequency \(\omega_{max}\) of the dispersion relation. Surely, the exact frequency limit can be obtained by replacing \(\kappa_{1,max}\) in (2.42), but it requires solving the complete analytical dispersion relation which is rather cumbersome. On the other hand, the maximum frequency \(\omega_{max}\) can be easily approximated using the effective modulus properties of the composite. In the effective modulus theory, a layered composite is assumed to behave as a transversely isotropic continuum as long as the layer thicknesses are sufficiently small compared to the wavelength of a harmonic excitation. The five independent effective elastic moduli corresponding to the considered layered composite can be obtained within the framework of statics \[152\] or dynamics \[157\]. Moreover, propagation velocities obtained from those effective elastic moduli characterize wave motion in the layered composite. However, the number of layers per unit cell should be carefully considered in order to suggest a FE model guideline in terms of the frequency \(\omega\) of propagating waves. In order to explore a relation between the wavenumber \(\kappa_1\) and the frequency \(\omega\) of propagation waves, the effective modulus theory for infinitely periodic multilayered composites \[12\] is adopted in this study.

Since the aliasing wavevector direction \(\kappa_1\)-axis corresponds to \(x_1\)-axis in the real lattice,
the frequency limit $\omega_{\text{max}}$ can be obtained from effective elastic property parallel to the layers. Moreover, the effective property of the shear wave is sufficient to find the limiting frequency $\omega_{\text{max}}$. Since the shear wave has smaller velocity than pressure wave \cite{180}, the lowest fictitious wave modes always comes from shear waves. Now, according to effective modulus theory, the relation between $\kappa_{1,\text{max}}$ and $\omega_{\text{max}}$ can be found as:

$$\omega_{\text{max}} = \bar{c}_s \kappa_{1,\text{max}}$$

(3.13)

where $\bar{c}_s$ is the effective shear wave velocity in the sagittal plane for wave propagation parallel to the layers and the bar sign $\bar{□}$ represents the effective properties of the multilayered periodic composites. The shear wave velocity $\bar{c}_s$ can be obtained from effective elastic modulus $\bar{c}_{44}$ and effective mass density $\bar{\rho}$ of the periodic composite by:

$$\bar{c}_s = \sqrt{\frac{\bar{c}_{44}}{\bar{\rho}}}$$

(3.14)

where

$$\bar{c}_{44} = \sum_{j=1}^{M} \left( \frac{n_j}{\mu_j} \right)^{-1} , \quad \bar{\rho} = \sum_{j=1}^{M} n_j \rho_j , \quad n_j = \frac{d_j}{\sum_{k=1}^{M} d_k} .$$

(3.15)

### 3.4.3 Guideline for Numerical Dispersion Relation

Being equipped with the anti-aliasing condition (3.12) and effective modulus based $\omega - \kappa_1$ relation (3.13), a generalized guideline can be provided for numerical dispersion relation. By substituting the approximate linear relation (3.13) into the anti-aliasing condition (3.12), the following generalized FE modeling guideline is obtained:

$$a_1 < \frac{2\pi \bar{c}_s}{\omega_{\text{max}} + \frac{\bar{c}_s \pi}{a_3} \tan \theta} .$$

(3.16)

Using (3.16), one can determine an adequate FE unit cell size $a_1$ of infinitely periodic multilayered composites for the highest frequency of interest $\omega_{\text{max}}$. The resultant equation (3.16) considers the number of layers per unit cell of infinitely periodic multilayered composites as well as the direction of oblique wave motion, which affect $\omega_{\text{max}}$. Alternatively, the accurate frequency range $\omega_{\text{max}}$ of the dispersion relation in FE framework for rectangular unit cell $a_1 \times a_3$ is determined
from (3.16):

\[ \omega_{max} < \bar{c}_s \left( \frac{2\pi}{a_1} - \frac{\pi}{a_3} \tan \theta \right) . \]  

(3.17)

The absolute guidelines presented by (3.16) and (3.17) is applicable to dispersion analysis in FE platform for all multilayered periodic composites and wave propagation at arbitrary directions. Note that, when wave propagates perpendicular to the layers (i.e., \( \theta = 0 \)), (3.16) and (3.17) reduce to:

\[ a_1 < \frac{2\pi \bar{c}_s}{\omega_{max}} \iff \omega_{max} < \frac{2\pi \bar{c}_s}{a_1} . \]  

(3.18)

The proposed guideline is evaluated for numerical dispersion relations which are affected by model aspect ratio and wave propagation angle. At first, consider the dispersion relations for wave propagation perpendicular to the layers of Sec. 3.3.2, where two IPMCs are analyzed for two aspect ratios. The Figs. 3.5D and 3.6D show the anti-aliasing guideline (3.17) by square marked black lines for aspect ratios \( a_1/a_3 = 2.0 \) and 1.0 corresponding to the 3-layered composite. Notice that the first fictitious modes which appear approximately at \( \bar{\omega} = 0.5 \) and 1.0 in Figs. 3.5D and 3.6D, are accurately predicted by the guideline. Similarly, the first fictitious modes in the numerical dispersion relation of the 4-layered composite captured in Figs. 3.7D and 3.8D. Therefore, the guideline (3.16) indicates the fictitious modes from spatial aliasing are located above the maximum valid frequency \( \omega_{max} \) for corresponding aspect ratio of the unit cell. On the other hand, the validity of the guideline (3.17) in the numerical dispersion relation for oblique wave motion can be illustrated using Figs. 3.9D and 3.10D. Notice that the lowest fictitious mode arising from front aliasing wavevector \( \Gamma'_{f} - \Lambda'_{f} \) lies above the suggested frequency limit \( \omega_{max} \) for wave propagation angles \( \theta = 45^\circ \) and \( 75^\circ \) in Figs. 3.9D and 3.10D, respectively. Regardless of the unit cell configuration and wave motion angle in IPMC, no fictitious modes are observed below the maximum valid frequency \( \omega_{max} \) suggested from (3.16).

3.5 Conclusion

The FE method offers an efficient framework to investigate the evolution of phononic crystals which possess materials or geometric nonlinearity subject to external loading. Despite its superior efficiency, the FE method suffers from spectral distortions in the dispersion analysis of infinitely
periodic multilayered composites. Spatial aliasing is identified as the long standing issue of the spectral distortion observed in numerical dispersion relations of layered composites where numerical techniques require the use of a rectangular unit cell in a 2-D coordinate space. The idea of spatial aliasing is validated using the analytical solution for the sagittal plane waves to reproduce fictitious modes induced from artificial finite periodicity in the direction parallel to the layers. Furthermore, combining an anti-aliasing condition and the effective elastic modulus theory, a generalized FE modeling guideline is provided to overcome the observed spectral distortions in FE dispersion relations of infinitely periodic multilayered composites. For a frequency range of interest, the suggested guideline can be adopted to obtain a valid FE dispersion relation for wave propagation at any direction in infinitely periodic multilayered composites.
4. DEVELOPMENT OF SHPB SYSTEM

4.1 Introduction

Verification from experimental studies is imperative for implementing the unique wave motion behavior of periodic layered composites in various practical applications. Specifically, the wave transmission property of the composites under finite deformation has not been thoroughly studied. Amplitude-dependent dispersion analysis of periodic structures is limited within weak nonlinearity \[107\] which doesn’t suffice in many cases. However, many of the practical applications require wave motion characteristics of these structures under finite deformation. A systematic experimental protocol can identify the finite deformation induced wave motion properties of periodic system. In order to perform the experimental studies on periodic layered composites, dynamic experimental systems are necessary. The electromagnetic shakers are commonly used to evaluate the transmission coefficient of periodic structure undergoing small amplitude \[41, 56, 78, 149, 163, 191, 197\]. This experiment can be considered as useful tool for validating the analytical studies. However, due to the limitation of excitation amplitude, the vibrations shakers are not sufficient to evaluate the attributions of finite deformation. Therefore, this study utilizes SHPB system to assess the wave transmission behavior under impact loading. This chapter briefly describes the development of viscoelastic SHPB system and also provides the configuration of electromagnetic shaker.

4.2 Development of SHPB

SHPB was primarily developed to determine the dynamic stress-strain relations of materials since the strain rates affect the mechanical properties of materials. The basic SHPB system consists of a striker, incident (or input) bar and transmitted (or output) bar as shown in Fig. 4.1. A specimen is placed between the incident bar and transmitted bar, while the striker is propelled using a pneumatic system. When the striker hits the input bar, a high impact force
is transferred into the input bar creating a compressive stress wave. As the wave propagates through the input bar, it reaches the interface of specimen and input bar. A fraction of the wave is reflected back in the input bar and the remaining part propagates through the output bar. Several strain gauges are placed on the input and output bar at different locations to capture the strain signals. From the analysis of these signals, necessary dynamic behavior of the specimen can be obtained. Besides the regular compressive SHPB, there have also been other basic types of SHPB such as tensile and torsional SHPBs.

Recently, the dynamic behavior of low impedance materials such as soft matter has attracted the attention of researchers. These low impedance materials do not show sufficiently significant strain signals when they are tested using metallic SHPB. Therefore, SHPB composed of low impedance viscoelastic bars is considered as an alternative solution in this study. In order to identify the amplitude-dependent wave motion behavior of periodic composites, an SHPB apparatus was built in the SEESL laboratory in University at Buffalo.

![Schematic of basic SHPB set-up showing input bar, output bar, striker and specimen.](image)

**Fig. 4.1:** Schematic of basic SHPB set-up showing input bar, output bar, striker and specimen.

### 4.2.1 Selection of Bar Material

The first step towards building a SHPB system is to determine the bar materials. For the experiments on high impedance material, 6061-T6 aluminum is selected. For low stiffness material testing, a wide range of polymeric materials are studied to assess their impedance at different strain rates for the viscoelastic SHPB. Since polymeric bars show significant viscoelasticity, the strain rate has significant effect on the stress-strain relationship (typically, Young’s modulus) of such materials.

For this study, a survey is made on the relative impedance of several polymers which is normalized by the cross sectional area. The term impedance refers to the ratio of applied force to resulting velocity. Table 4.1 shows the comparison of impedance for the polymeric materials. Nylon-66, UHMWPE and PTFE were selected out of all the polymers for further study based
on the impedance distribution between them. Then, a literature review is conducted to obtain the stress-strain curves of these polymers at low and high strain rates (approximately up to 1000/sec). Figs. 4.2 and 4.3 show the comparison of the stress-strain relations at low and high strain rates, along with the initial elastic modulus. As mentioned earlier, the elastic modulus varies with strain rates for all the materials due to viscoelastic effect. However, it is found that the initial elastic modulus of Nylon-66 is not significantly affected by strain rates in comparison to other two polymeric materials. Therefore, Nylon-66 is selected for the polymeric SHPB.

**Tab. 4.1: Impedance Comparison of Polymeric Materials**

<table>
<thead>
<tr>
<th>Polymeric Material</th>
<th>Modulus [Nm$^{-2}$]</th>
<th>Density [kgm$^{-3}$]</th>
<th>Phase Velocity [ms$^{-1}$]</th>
<th>Impedance/Area [kgm$^{-2}$s$^{-1}$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>PTFE</td>
<td>$4.6 \times 10^8$</td>
<td>2200</td>
<td>457.26</td>
<td>$1.01 \times 10^6$</td>
</tr>
<tr>
<td>LDPE</td>
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<td>920</td>
<td>634.17</td>
<td>$5.83 \times 10^5$</td>
</tr>
<tr>
<td>UHMWPE</td>
<td>$5.5 \times 10^8$</td>
<td>950</td>
<td>760.89</td>
<td>$7.23 \times 10^5$</td>
</tr>
<tr>
<td>HDPE</td>
<td>$7.0 \times 10^8$</td>
<td>950</td>
<td>858.40</td>
<td>$8.15 \times 10^5$</td>
</tr>
<tr>
<td>Nylon 66</td>
<td>$1.5 \times 10^9$</td>
<td>1140</td>
<td>1147.08</td>
<td>$1.31 \times 10^6$</td>
</tr>
<tr>
<td>ABS</td>
<td>$1.4 \times 10^9$</td>
<td>1040</td>
<td>1160.24</td>
<td>$1.21 \times 10^6$</td>
</tr>
<tr>
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<tr>
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<td>1020</td>
<td>1565.56</td>
<td>$2.13 \times 10^6$</td>
</tr>
</tbody>
</table>

**Fig. 4.2: Comparison of stress-strain relations of polymers at low strain rate**
4.2.2 General Configuration of SHPB System

The schematic of a basic SHPB system in Fig. 4.1 shows the input bar, output bar and striker alongside the characteristics strain signals. The striker launcher system, which is not presented in this figure, is discussed in detail later in the Sec. 4.2.3. Fig. 4.4 shows the metallic and polymeric SHPB systems in the SEESL lab which can be operated using same the striker launcher. The entire system is supported on four sections of vertically-placed I-beams at the base. These base beams support two horizontally-placed I-beams that carry one SHPB each. Shaft supports are used to hold the input and output bars whose locations are determined by performing the buckling analysis of bars. The maximum expected forces in the bars are obtained from an ABAQUS analysis using Nylon-66 material properties.
4. Development of SHPB System

4.2.2.1 Bar Diameter

The diameter of the Hopkinson bar is one of the important factors that determine the applicability of 1D wave propagation theory. One critical assumption of 1-D wave theory is that no wave propagation is allowed in the direction of the bar diameter. In order to generate one dimensional plane wave motion in SHPB bars, the length-to-diameter ratio should be larger than 20 \cite{45}. However, the ratio is far larger in practice and it is governed by the intended specimen dimension. Since the present study is planned for phononic crystals, the specimen dimension is considered to be approximately 45\,mm. As a result, the diameter of the SHPB bars is selected to be 50\,mm which is larger than the potential specimen size. Consequently, the SHPB is designed with input and output bars of 3.0\,m long which gives a length to diameter ratio of \((L_{\text{Bar}}/D_{\text{Bar}} = 60)\).

4.2.2.2 Striker Length

The time interval between the arrival of incident and reflected signals in a strain gage located on the input bar is dependent of the striker length by \(\Delta t = 2L_{\text{str}}/c_{0,\text{str}}\), where \(c_{0,\text{str}}\) is the striker bar velocity. For the accurately distinguishing the incident and reflected waves in the input bar, the striker length is typically suggested to be less than half of the input bar for conventional SHPB \((2L_{\text{str}} < 0.5L_{\text{in}})\). But, generally for large diameter SHPB bars similar to the present case, the duration of time interval \(\Delta t\) is smaller which requires significantly shorter striker. However, in the event of overlapping waves, the frequency domain analysis can be performed to separate the incident and reflected waves \cite{170}.

4.2.2.3 Striker Guide

When a striker is propelled using a launching system, it has to be guided so that it hits the input bar concentrically. This requirement can be ensured by the use of a striker guide. In order to keep provision for future experiments which might require long strain signals, the striker guide is designed to operate 1.8\,m long strikers. Since striker guide has to be longer than the maximum striker length, a 2.5\,m long striker guide is set up for the SHPB. Note that when a striker travels inside the striker guide, it experiences friction with the inner wall of the guide. For minimizing friction, the strikers are fit inside circular rings so that the contact area between the striker and the guide become minimal. In search for material with low density and friction coefficient, UHMW is selected due to its low friction coefficients with metals. In addition, the
pressure durability of the guide should be taken into consideration. Since the striker is launched using a pneumatic system, the striker guide may be subjected to high pressure in the range of 3.5 $MPa$. For this reason, a heavy duty steel pipe, namely, the seamless 316L schedule 80 pipe is selected. The 76 $mm$ nominal size steel pipe have an inner diameter of 74 $mm$ with pressure rating of more than 14 $MPa$.

4.2.3 Striker Launcher

The striker launching system performs a vital operation of the SHPB by propelling striker towards the input bar. Generally, pneumatically activated striker launchers are preferred for its safe and simple maneuvering technique. This section describes the systematic development of striker launcher.

4.2.3.1 Required Pressure

The preliminary step for designing a pneumatic striker launcher system is to determine the pressure required for propelling various sizes of strikes. While calculating the pressure requirement, the essential factor to consider is the air-tightness of the striker launcher system. For a properly air-tight launcher system, the pressure does not drop and the acceleration of the striker remains constant. On the other hand, if there is a leak in the pipe fittings, pressure drops with time and thereby causing a varying acceleration. These two scenarios are considered in the design of the launch system. Four striker sizes are selected to determine the pressure requirement in both the constant and varying acceleration cases. Figs. 4.5A and 4.5B show the relationship between air pressure and striker velocity for different sizes of nylon strikers for constant and varying acceleration condition, respectively.

4.2.3.2 Design of Launcher System

It is observed in Fig. 4.5 that significant pressure difference exists between the two cases. However, the constant pressure to propel striker represents an ideal condition which can be rarely found in actual experimental systems. Therefore, the pneumatic design of the launcher is based on the varying acceleration requirement. The projected maximum speed for the experiments was 20 m/s. The heaviest nylon striker would require approximately 1.8 $MPa$ pressure. Therefore, the entire launcher system is designed to generate 3.5 $MPa$ pressure on the striker. The supply gas in the
lab was available at approximately 1 $MPa$ pressure. Since the supply pressure is inadequate for the SHPB experiment, high pressure compressed gas cylinder is considered for striker launcher. The system undertaken for the launcher included two tanks along with solenoid valve for each SHPB apparatus. Cylinder tanks with compressed nitrogen gas at pressure as high as 17 $MPa$ is readily available from suppliers. From this cylinder tank, the gas had to be regulated down to a secondary cylinder tank at a required pressure and combine it with a solenoid valve which is operated electrically. The overall striker launcher shown in Fig. 4.6 can be divided into following parts:

1. First cylinder tank with compressed nitrogen gas at 17 $MPa$
2. Regulator to control gas flow between two cylinders
3. Second cylinder tank containing reduced pressure gas
4. Pipe connecting two cylinders
5. A T-section pipe referred as 1$^{st}$ line which distributes gas from 2$^{nd}$ cylinder to solenoid valves
6. 2$^{nd}$ line of straight pipe which connects solenoid valve to inlet of striker guide
4.2.3.3 Equipment selection

- **Gas Cylinder and Regulator**

The first cylinder tank with higher pressured gas requires replacement after several tests. However, the second cylinder tank which is smaller in size is attached to the fixed fixture of the SHPB. A G-size cylinder tank is procured from Praxair, Inc.

The regulator needed to control the gas pressure was also bought from Praxair. A high capacity regulator was selected with 20.5 MPa inlet pressure (connected with the large cylinder tank) and 10.25 MPa outlet pressure (connected with the small cylinder tank). For pipe fittings, 6.35 mm pipes are selected for the gas cylinder tanks. So, on the outlet of the regulator has 6.35 mm NPT (National Pipe Thread). In addition, CGA 580 connector is recommended for nitrogen gas cylinder. Thus, the regulator had a CGA 580 connector at the inlet end, and it also contained a needle valve at the outlet end for safety purpose. Similarly, another CGA 580 connector was needed for the input of small cylinder.

The outlet gage of the regulator is replaced by a digital gauge from Omega Engineering Inc which provides a minimum resolution of 0.007 MPa with a maximum capacity of 7 MPa.

- **Solenoid Valve**

For the impact test using SHPB, the striker has to be operated using a pneumatic system where a sudden release of compressed gas is indispensable. In the medium strain rate ranges, hand valves are popular for SHPB. However, the use of hand valve does not give desired control of the striker velocity.

In search for a suitable pneumatic system, it became apparent that for a sudden release and cease of compressed gas, solenoid valve is the most appropriate option. There are several
controlling factors in determining the configuration of the solenoid valve, such as flow rate, orifice size and pressure limit. The conventional solenoid valve available in the industry can provide either high pressure capacity or high flow rate capacity. After an extensive search for solenoid valve, an Atkomatic normally closed valve is selected for the SHPB. Although the launcher system is planned to work in air-tight condition, in practice various sources can cause some loss in air pressure. Fig. 4.5 shows that to obtain 20 m/s velocity of 1.8 m long Nylon striker, the required pressure can be between 1.5 MPa or 0.5 MPa depending on the air-tight condition. A sample calculation is presented here for the Atkomatic141A66 normally closed valve with 38 mm orifice size. By considering an inlet pressure of 1.4 MPa, the flow rate equation which is provided by the supplier can be evaluated in SCFH unit (standard cubic feet per hour) as:

\[ F_r = 1360C_v \sqrt{\frac{P_1\Delta}{ST}} \]  

(4.1)

where, \( T = 530^\circ R \) is the absolute temperature in Rankine; \( P_1 = 1500 \text{kPa} \) and \( P_2 = 1273 \text{kPa} \) are inlet and outlet pressures, respectively; \( \Delta P = 227.0 \text{kPa} \) is the pressure drop; \( S = 0.967 \) is the specific gravity at 60\(^\circ\)F and 0.1013 MPa; and \( C_v = 25.0 \) is the valve Flow coefficient. Now, the flow rate can be found as

\[ F_r = 1360 \times 25 \sqrt{\frac{P_0 \times 215}{0.967 \times 530}} = 120616.74 \text{ SCFH} \]  

(4.2)

Flow rate at actual pressure and temperature (i.e., ACFH) is:

\[ F_r = 120616.74 \text{SCFH} \frac{P_{std}}{P_{actual}} \frac{T_{actual}}{T_{std}} \]

\[ = 120616.74 \frac{14.7(460 + 70)}{185(460 + 32)} = 10324.38 \text{ cfh} = 292.35 \text{ m}^3\text{h}^{-1} \]  

(4.3)

Using on-line program named *Peace Software* the following properties are obtained for nitrogen gas at 1.273 MPa. The properties are shown in Table 4.2.
Tab. 4.2: Properties of nitrogen at 185 psi from software

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Medium</td>
<td>Nitrogen</td>
<td></td>
</tr>
<tr>
<td>Pressure</td>
<td>12.75</td>
<td>[bar]</td>
</tr>
<tr>
<td>Temperature</td>
<td>21.11</td>
<td>[Celsius]</td>
</tr>
<tr>
<td>Density</td>
<td>14.66</td>
<td>[kg m(^{-3})]</td>
</tr>
<tr>
<td>Specific Enthalpy</td>
<td>302.55</td>
<td>[kJ kg(^{-1})]</td>
</tr>
<tr>
<td>Specific Entropy</td>
<td>6.08</td>
<td>[kJ kg(^{-1})K(^{-1})]</td>
</tr>
<tr>
<td>Specific isobar heat capacity</td>
<td>1.06</td>
<td>[kJ kg(^{-1})K(^{-1})]</td>
</tr>
<tr>
<td>Specific isochor heat capacity</td>
<td>0.75</td>
<td>[kJ kg(^{-1})K(^{-1})]</td>
</tr>
<tr>
<td>Isobar coefficient of thermal expansion</td>
<td>3.51</td>
<td>[10(^{-3}) k(^{-1})]</td>
</tr>
<tr>
<td>Heat conductance</td>
<td>26.27</td>
<td>[10(^{-3}) Wm(^{-1})k(^{-1})]</td>
</tr>
<tr>
<td>Dynamic viscosity</td>
<td>17.81</td>
<td>[10(^{-6})Pa s]</td>
</tr>
<tr>
<td>Kinematic viscosity</td>
<td>1.34</td>
<td>[10(^{-6})m(^2) s(^{-1})]</td>
</tr>
<tr>
<td>Thermal diffusivity</td>
<td>18.60</td>
<td>[10(^{-7})m(^2) s(^{-1})]</td>
</tr>
<tr>
<td>Coefficient of compressibility</td>
<td>0.99</td>
<td></td>
</tr>
<tr>
<td>speed of sound, (v_a c_p)</td>
<td>351.57</td>
<td>[m s(^{-1})]</td>
</tr>
</tbody>
</table>

The general incompressible fluid formulas are not applicable in the current situation. Therefore, the compressibility of the gas should be determined. Table 4.2 shows that the speed of dilatational wave in nitrogen gas is \(v_a = 351.57 m s^{-1}\). Also notice that \(P_2/P_1 > 0.53\) which indicates a subcritical flow condition. The speed of gas at the 38mm outlet can be determined using the suction surge velocity:

\[
V_{suc} = \frac{\Delta P}{\rho v_a} = 287.21 m/s
\]

\[
Mach number = \frac{\text{Speed of gas flow}}{\text{Speed of sound}} = 0.81 > 0.3
\] (4.4)

where the pressure difference is the difference between inlet and outlet pressure before opening. Since Mach number is greater than 0.3, the flow is in subsonic range but compressible. Since it is not in the sonic range, a normal shock wave will not be generated and isentropic condition can be considered.
In order to obtain the pressure drop, the Mach number at the second section is calculated using the area ratio:

\[
\frac{A_2}{A_1} = \frac{M_1}{M_2} \left( \frac{2 + (k_m - 1)M_2^2}{2 + (k_m - 1)M_1^2} \right)^{\frac{k_m}{2(k_m - 1)}}
\]

which provides \(M_2 = 0.14\). By employing the Mach number at second section, the pressure at that section can be determined through the pressure ratio:

\[
\frac{P_1}{P_2} = \left( \frac{2 + (k_m - 1)M_2^2}{2 + (k_m - 1)M_1^2} \right)^{\frac{k_m}{2(k_m - 1)}}
\]

Through (4.6), the pressure at section 2 is found to be \(P_2 = 1916.74kPa\). Then, the velocity of the flow in an isentropic condition can be determined from:

\[
\Rightarrow \int_{v_1}^{v_2} \frac{dv}{v} = -\frac{1}{(1 - M_2^2)} \int_{A_1}^{A_2} \frac{dA}{A},
\]

which results in \(v_2 = 9.38\ m/s\) and indicates that the order of velocity is in a diffuser condition. Since the Mach number at section 2 is 0.179, an incompressible flow condition can be considered to find the velocity:

\[
v_2 = \frac{Q}{A_2} = \frac{292.35/3600}{4.562 \times 10^{-3}} = 17.8m\ s^{-1}
\]

So, the results from both equations indicate same order of magnitude. Therefore, final pressure and velocity at the Striker Guide inlet is found as:

\[
P_2 = 1916.74kPa
\]

\[
v_2 = 9.38 - 17.8\ m/s
\]

Recall that the desired velocity of the largest nylon striker is \(20\ m/s\). The velocity range obtained from (4.9) indicates that the valve model Atkomatic141A66 has the capacity to launch the largest nylon striker at required velocity.
4.2.4 Accompanying Accessories

- **Digital timer**

Although an ideal striker launcher system should be able to maintain constant pressure on the striker, this is often impossible to achievable in practice. Therefore, the compressed gas pressure and striker velocity relation generally lies between previously mentioned constant and varying acceleration conditions. Moreover, the transition of gas pressure between the solenoid valve and striker guide shows that the compressibility of the gas changes abruptly. Hence, a better controlling system is needed for the driving the striker at desired speed. One way to improve the gas flow through the solenoid valve was controlling the duration of open valve. The selected Atkomatic solenoid valve which is operated by AC power can be kept open for a certain duration by controlling the power supply. The exact time needed to open and close the valve is approximately known. The data from the manufacturer showed that the opening and closing time for the valve are $50 \text{ msec}$ and $170 \text{ msec}$, respectively.

In order to control the gas flow through solenoid valve, a digital timer with millisecond range resolution should be selected. The timer allows finding a relation between open valve duration and gas pressure with the striker speed for different types of striker. The selected OMRON H5CX digital timer has minimum resolution of $1 \text{ msec}$. The small resolution enables in managing the duration of gas flow and eventually provides the opportunity to control the striker velocity.

- **Pipes and Fittings**

Since the experiment deals with high pressure in the striker launcher system, the pipes and fittings were selected to withstand the condition. It was found that the Schedule 80 pipes have pressure ratings above $1.4 \text{ MPa}$, and therefore they would suit the experiment. All the Schedule 80 steel pipes and fittings were obtained from Commercial Pipe and Supply Corporation.

The first cylinder tank needs repeated replacement whenever the gas pressure becomes low. Thus, the connection between the regulator and the first cylinder tank was set by a synflex hose because it has the flexibility that allows the ease of replacement and also the alignment of new cylinder becomes safe. A synflex hose with $20 \text{ MPa}$ pressure rating was purchased from Power Drives.

- **Bar Support**
The bar support system shown in consists of eight support for each of the input and output bar. These supports are also called shaft support and have 50mm inside diameter. The shaft support used for the SHPB is made up of two pieces. The detachable top part allows easy replacement of new bar.

- **Velocity sensor**

The velocity of the striker just before striking the input bar is an important data for calibrating the experiment. In the present SHPB machine, the striker velocity is measured using non-contact photoelectric sensor, which detects a change in light intensity. Two sets of sensors are used for the measurement. Each sensor system contains a sensing device with two fiber optic cables. The fiber optic cables work as light guide, where one acts as light source and another works as light receiver. The light source continuously transmits a light beam to the receiver. When the striker pass through the detection path of the cable set, the light beam is broken and the signal can be captured. To measure the striker velocity, two set of such sensors were placed at a known distance apart. The striker passing through these two sensors gives the time of passing for each sensor. When the time difference is known from two sensors, the velocity of the striker can be calculated. Fig. 4.8 shows the set-up of photoelectric sensor used for determining the velocity of striker.

![Laser based photoelectric sensor for measuring striker velocity](image)

- **Strain gage**

Multiple strain readings are required for the experiments. Linear foil strain gages are generally used for metallic SHPB tests. In case of polymer bars, the strain is relatively higher. In order to accommodate the strain of polymeric bars, high strain capacity of the strain gage is expected.
4. Development of SHPB System

Vishay strain gage *EP Series* has high strain range as much as 10%. So, for the experiment with polymeric bar, *EP-08-031DE-120* strain gage was selected. The gauge has (0.78 mm) gage length and 120 Ω resistance. On the other hand, ‘EA Series’ strain gages, which has 3% elongation capacity was used for metal SHPB. The signal conditioning amplifiers used for amplifying the strain signals are Vishay 2310 Signal Conditioning Amplifier.

5

Data acquisition system

• Data acquisition system

The high strain rate impact test in SHPB demands high sampling rate of data for proper calculation. In the current SHPB, a high capacity DAQ system from National Instruments, *NI PCI-6115 S* Series was installed. This system performs simultaneous data processing at each of the channel. The DAQ can give 12 bit output at 10 MHz per channel sampling rates. In addition, this system was integrated with LabVIEW software for signal data collection and processing.

4.3 Calibration of SHPB

The primary objective of employing the SHPB system is to determine the transmission coefficient of elastic-viscoelastic composites. Since, the viscoelastic constituent of the composite have significantly low impedance compared to traditional metallic elastic materials, the signal to noise ratio becomes a concern during the SHPB experiment.

Since conventional metallic SHPB systems has high impedance mismatch with low stiffness materials, polymeric SPHB systems with low impedance are adopted [13, 170] However, the available input force is limited from polymeric SHPB which can impede the amplitude-dependent study. Since the current study focuses on the experimental evaluation of periodic layered composite which is made of elastic and viscoelastic layers, both polymeric and metallic SHPB system should be considered. Beforehand, the experimental technique should be developed through necessary calibrations for proper operation using the SHPB system. Therefore, a complete calibration is conducted on the nylon and alunminum SHPB systems of SEESL laboratory in several steps, which are described in this section.
4.3.1 Bar Alignment Operation

For an ideal SHPB experiment, the striker, input and output bar should be properly aligned. However, various sources of misalignment can cause considerable distortion of strain signal. The primary reasons of bar misalignment in SHPB system and the corresponding preventive measure are discussed below:

- Uneven base or support system

Uneven support system can cause eccentricity of the longitudinal axes of the input and output bars. Also, the misalignment between the striker guide can cause off-center impact between striker and the input bar. These issues can be avoided by using a straight platform and bar support system. The SHPB in SEESL lab is installed on a single I-beam section which provides a uniform plane for the entire SHPB system. In order to support the input and output bars, low tolerance aluminum shaft supports are used. Furthermore, alignment of all the shaft supports and the striker guide is adjusted using high precision laser beam. In addition, coaxial impact of striker to the input bar is confirmed by using a small tapered section at the striker end as shown in Fig. 4.9. Therefore, the contribution of misalignment from the SHPB support system has been significantly minimized.

![Fig. 4.9: (A) Top view and (B) front view of tapered nylon striker](image)

- Straightness tolerance of bars

Both metal and polymer industrial bars have a straightness tolerance during manufacturing. However, the straightness of polymeric materials is further affected by additional sources, such
as temperature, moisture content, creep, etc. Specially, due to viscoelastic nature of the nylon bar, straightness is primarily impaired by creep effect. The creep effect is minimized by using densely spaced bar support system. Both the input and output bars are supported by eight shaft supports at a spacing of 0.375 m. Nonuniform straightness induces flexural modes in the bars even with good support system and coaxial impact by striker. Since the SHPB apparatus is based on one dimensional longitudinal wave theory, the appearance of flexural mode can produce inaccurate results. In the current SHPB system, the flexural mode is avoided by using the average value of a pair of strain gauges. Specifically, at the intended strain measurement location, two strain gages are placed on opposite radial sides of the bar. As a result, the compression and tension flexural waves of two sides nullify each other and the averaged signal only contains the stress due to longitudinal wave. Fig. 4.10A shows a pair of typical strain signals at top and bottom faces of nylon bar which consist of both longitudinal and flexural wave modes. The averaged longitudinal strain signal in Fig. 4.10B represents the purely longitudinal strain measurement which is devoid of flexural modes. Similar approach is also followed in during the measurement using aluminum SHPB.

![Fig. 4.10: (A) Strain signals from radially opposite sides of a SHPB bar (B) averaged signal at the same location representing only the strain due to longitudinal wave](image)

- **Irregular end face of input and output bars**

The circular end face of the input and output bars should have straight plane and be parallel to each other. A concave or convex face in either bar can give rise to erroneous reflected and transmitted signal. The straightness of the planes can be checked by placing input and output
bars in direct contact of each other so that they form a continuous homogeneous bar. If the end
planes of both bars are straight and parallel, a strain signal from two bars should be same at the
contact interface.

4.3.2 Propagation Coefficient Calibration

It is well established that wave propagation through metallic and polymeric SHPB systems can
undergo substantial distortion due to wave dispersion and attenuation [13, 170]. Therefore, both
aluminum and nylon SHPBs developed in the SEESL facility should be calibrated to determine
frequency dependent propagation coefficient.

4.3.2.1 Single Bar Impact Test for Individual Striker

The polymeric bars are governed by inherent damping due to viscoelastic nature of the material.
As a result, the general experimental formulation of elastic SHPB system cannot be employed.
Specifically, the experimental measurements (i.e., strain, displacement etc.) taken on the arbitrary
locations of the input and output bars can not be directly used as the mechanical signal at the
interfaces between specimen and SHPB bars. Furthermore, previous studies [170] have shown
that the wave dispersion can have significant effect on the metallic SHPB system. The content of
the stress waves should be modified to incorporate the wave dispersion and attenuation behavior
of both SHPB systems during the analysis. Therefore the current study conducts calibration for
wave propagation property of the nylon and aluminum bars. In order to determine wave motion
behavior of the bar material, the time domain signal is analyzed in the frequency domain. In
this approach, the strain signals are converted to frequency domain by Fourier transformation
which is composed of rightward traveling wave \( \hat{\varepsilon}_R(x_o, \omega) \) and leftward traveling wave \( \hat{\varepsilon}_L(x_o, \omega) \)
with respect to reference point \( x_o \):

\[
\hat{\varepsilon}(x, \omega) = \hat{\varepsilon}_R(x_o, \omega)e^{-i\kappa_{\text{bar}}(\omega)(x-x_o)} + \hat{\varepsilon}_L(x_o, \omega)e^{i\kappa_{\text{bar}}(\omega)(x-x_o)}
\]  (4.10)

where \( \kappa_{\text{bar}}(\omega) \) is the propagation coefficient of the SHPB bar obtained using complex modulus
\( \tilde{E}(\omega) \) as:

\[
\kappa_{\text{bar}}^2(\omega) = \frac{\rho\omega^2}{\tilde{E}(\omega)} .
\]  (4.11)
The propagation coefficient $\kappa_{\text{bar}}(\omega)$ is complex-valued which is represented by:

$$\kappa_{\text{bar}}(\omega) = \kappa_{\text{R bar}}(\omega) + i \kappa_{\text{I bar}}$$

(4.12)

where $\kappa_{\text{R bar}}$ and $\kappa_{\text{I bar}}$ are wavenumber and attenuation coefficient, respectively. The characterization operation of $\kappa_{\text{bar}}(\omega)$ is referred as single-bar impact test. This procedure utilizes frequency spectra of incident wave $\dot{\varepsilon}_{\text{inc}}$ and reflected wave $\dot{\varepsilon}_{\text{ref}}$ from free end which are recorded at same location to obtain transfer function $H(\omega)$ [13] as:

$$H(\omega) = -\frac{\dot{\varepsilon}_{\text{ref}}}{\dot{\varepsilon}_{\text{inc}}} = e^{-2i\kappa_{\text{bar}}(\omega) d}$$

(4.13)

where $d$ denotes the distance of the strain gauge from free end. Then, the transfer function $H(\omega)$ is used to calculate components of propagation coefficients:

$$\kappa_{\text{R bar}}(\omega) = -\frac{\arg|H(\omega)|}{2d} \quad \text{and} \quad \kappa_{\text{I bar}}(\omega) = \frac{\ln|H(\omega)|}{2d}.$$  

(4.14)

Using the information of $\kappa_{\text{R bar}}(\omega)$ and $\kappa_{\text{I bar}}(\omega)$, the measured strain signals can be transported using (4.10) from input and output bar strain gauge locations to the interfaces between specimen-input bar and specimen-output bar, respectively.

![Fig. 4.11:](image)

Fig. 4.11: (A) Incident and reflected strain readings of single bar experiment by a 250 mm striker and the corresponding (B) dispersion relation between phase velocity and frequency, and (C) frequency dependent attenuation coefficient

Note that the frequency range of valid propagation coefficient $\kappa_{\text{bar}}(\omega)$ is governed by the input energy and shape of pulse. Therefore, the equipment is calibrated for all the striker sizes
where striking velocity is varied for each case. In order to identify the valid frequency range of experimentally calculated \( \kappa_{\text{bar}}(\omega) \), the longitudinal phase velocity \( c_l \) (i.e., \( \kappa/\omega \)) is compared with the Pochhammer-Chree longitudinal wave equation \([67]\) of circular bar:

\[
\frac{2\alpha}{r_b}(\beta^2 + \kappa^2)J_1(\alpha r_b)J_1(\beta r_b) - (\beta^2 - \kappa^2)^2 J_0(\alpha r_b)J_1(\beta r_b) - 4\kappa^2 \alpha \beta J_1(\alpha r_b)J_0(\beta r_b) = 0 \quad (4.15)
\]

where \( \alpha \) and \( \beta \) are pressure and shear wave related components, \( r_b \) is the bar radius and \( J_0 \) and \( J_1 \) are Bessel function of first kind. Fig. 4.11 represents a typical calibration result of nylon SHPB which is obtained from single bar experiment for 250 mm long striker. For instance, Fig. 4.11A presents the actual incident and reflected wave generated by the 250 mm striker at 5.09 m/s velocity and the corresponding phase velocity and attenuation coefficient are shown in Figs. 4.11B and 4.11C, respectively. Notice that the experimental phase velocity in Fig. 4.11B shows good resemblance with the analytical solution (4.15) upto 12 kHz. In addition, the attenuation coefficient of Fig. 4.11C display approximately linear relation with frequency within the similar range.

### 4.3.2.2 Refined Propagation Coefficient

It is clear from the results of individual propagation coefficients that the phase velocity and attenuation coefficients are dependent on the striker size and velocity. Moreover, the propagation coefficient is affected by pulse duration produced by the striker which causes spike pattern in the spectra of phase velocity and attenuation coefficient. Nevertheless, a propagation coefficient spectra should be chosen for each SHPB to transfer a signal from the stain measurement location to the interface of bar and specimen. Instead of relying on propagation coefficient values of a single test, an averaged scheme is adopted in the current study. Specifically, propagation coefficient of both aluminum and nylon bars are established from 30 single-bar tests to obtain averaged values of phase velocity and attenuation coefficient. Fig. 4.12 represents the average propagation coefficient spectra by solid lines and corresponding one standard deviation by shaded areas for aluminum and nylon bars. In addition, the propagation coefficient results are also compared with the analytical solution of (4.15). Although, the relation (4.15) is developed for elastic bar, it serves as an effective indicator of attainable frequency range of any SHPB system. For instance, Figs. 4.12A and 4.12C show the comparison between analytical and experimental
phase velocity spectra of aluminum and nylon bars, respectively. It can be noticed in Figs. 4.12A and 4.12C that, the average values of experimental phase velocity (solid lines) have acceptable resemblance with the analytical results (dashed lines) up to 45 kHz and 12 kHz for aluminum and nylon bars, respectively. Note that, the calibrated phase velocity of nylon bar deviate from the elastic analytical solution at low frequency due to viscoelastic effect of the material. In addition, the attenuation coefficients of Figs. 4.12B and 4.12D display approximately linear relation within the valid frequency limit for corresponding bars. Therefore, proposed SHPB configuration consisting of aluminum and nylon bars is adequate for studying the transmission spectra of a specimen up to 12 kHz.

![Fig. 4.12: (A) Phase velocity and (B) attenuation coefficient representing the propagation coefficient of aluminum bar. Propagation coefficient of nylon bar showing (C) phase velocity and (D) attenuation coefficient. The average experimental results are denoted by solid lines and standard deviations are represented by shaded areas. In addition, analytical solution is shown by dashed line in phase velocity spectra.](image-url)
4.3.3 SHPB Bar Alignment Calibration

Generally, the large diameter metallic and polymeric bars procured from industry do not contain plane and straight end face. Ideally, a face straightening operation should be carried out in the laboratory. However, the regular lathe machines are unable to maneuver the 10 ft (3 m) long nylon and aluminum bars. Hence, the present study has considered a new approach for face alignment. In this simple procedure, two fine grained sand papers were used for sanding the input and output bar end faces simultaneously. The proper alignment of the bars is marked to keep track of the uniform contact setting.

The calibration for uniform contact can be performed by transferring the input and output signals to the contact location from input and output bars, respectively. Consider the strain signals $\varepsilon_{R,inp}(x_{inp}, t)$ and $\varepsilon_{R,out}(x_{out}, t)$ are measured at input and output bar locations $x_{inp}$ and $x_{out}$, respectively. Note that, both the signals represent rightward travelling wave. In frequency domain, the input bar strain $\hat{\varepsilon}_i(x_c, \omega)$ and output bar strain $\hat{\varepsilon}_o(x_c, \omega)$ at contact location $x_c$ can be obtained by pushing forward the input bar strain $\hat{\varepsilon}_{R,inp}(x_{inp}, \omega)$ and pulling backward the output bar strain $\hat{\varepsilon}_{R,out}(x_{out}, \omega)$, respectively:

$$
\hat{\varepsilon}_i(x_c, \omega) = \hat{\varepsilon}_{R,inp}(x_{inp}, \omega)e^{-i\kappa_{bar}(\omega)(x_c - x_{inp})}
$$

$$
\hat{\varepsilon}_o(x_c, \omega) = \hat{\varepsilon}_{R,out}(x_{out}, \omega)e^{i\kappa_{bar}(\omega)(x_{out} - x_c)}.
$$

(4.16)

The final strain signals $\varepsilon_i(x_c, t)$ and $\varepsilon_o(x_c, t)$ can be obtained from inverse Fourier transform of $\hat{\varepsilon}_i(x_c, \omega)$ and $\hat{\varepsilon}_o(x_c, \omega)$, respectively. For perfectly plane contact, the signals $\varepsilon_i(x_c, t)$ and $\varepsilon_o(x_c, t)$ should overlap each other. Typical results of uniform contact calibration of nylon SHPB are shown in Fig. 4.13 for four different striker sizes. The input and output signals obtained from (4.16) are shown by solid and dashed lines, respectively. The good agreement between the signals indicates proper alignment of the end faces of input and output bars. Furthermore, this calibration proves the accuracy of the propagation coefficient spectrum $\kappa_{bar}(\omega)$. 
Fig. 4.13: Transferred input and output strain signals at the contact interface generated by (A) 150 mm striker with impact velocity 4.47 m/s, (B) 200 mm striker with impact velocity 4.71 m/s, (C) 250 mm striker with impact velocity 4.75 m/s and (D) 300 mm striker with impact velocity 4.48 m/s.
5. AMPLITUDE-DEPENDENT WAVE IN PERIODIC VISCOELASTIC COMPOSITE

5.1 Introduction

Phononic crystals are periodic structures designed to control mechanical waves through Bragg scattering [90, 173]. Under harmonic excitations with infinitesimal deformation, analyses show that they can possess intriguing wave characteristics such as phononic band-gaps [22, 28, 40, 108], acoustic diodes [99, 100, 104], and acoustic waveguide [158]. Many experimental studies have been conducted to evaluate wave transmission properties of various phononic crystals as well. Using electrodynamic shakers [41, 149, 197] or piezoelectric actuators [76, 108, 154], researchers typically apply infinitesimal deformations or small amplitude forces to investigate the wave characteristics of linearly perturbed phononic crystals.

Recently, there has been a rising interest in the nonlinear wave transmission behavior of phononic crystals. There is a wide range of numerical studies of nonlinear phononic crystals. For instance, some researchers introduce nonlinear constitutive relations in discrete lattice models and investigate the evolution of the dispersion relations [106, 128]. In addition, the wave dispersion relation of weakly nonlinear periodic structures is studied in the finite element (FE) framework [129]. The nonlinear characteristics of wave motion in phononic crystals have also been numerically studied by exploring solitary waves in 1-D granular crystals [5, 63]. However, experimental studies regarding the nonlinear wave transmission behavior of phononic crystals have been predominantly performed on only 1-D granular crystals [53, 54, 81]. In these experiments, customized impact apparatus is employed to generate solitary waves in a 1-D chain of elastic beads, whose nonlinearity mainly originates from Hertzian contacts. Then, measured solitary waves are analyzed to identify the impulse-dependent wave transmission characteristics [198] or tunable phononic band-gaps [22].

This study experimentally investigates the nonlinear wave transmission behavior of a con-
tinuum phononic crystal. Note that continuum phononic crystals are commonly composed of metals and polymers to exploit their high impedance mismatch which brings down the intriguing frequencies to the acoustic frequency range \([109, 122]\). Due to the inherent viscoelastic and damping properties of polymers, electrodynamic shakers or piezoelectric actuators are not applicable to generate sufficiently large excitation to induce nonlinear wave motion in continuum phononic crystals containing polymers. Here, the split Hopkinson pressure bar (SHPB) apparatus is employed as a tool to identify the nonlinear wave characteristics of 1-D continuum viscoelastic phononic crystals. At first, this chapter presents the details of a 1-D viscoelastic phononic crystal under investigation in Sec. 5.2. Then, Sec. 5.3 discusses the impulse-independent wave transmission characteristics of the viscoelastic phononic crystal by reviewing its analytical dispersions relation and performing base excitation tests with an electrodynamic shaker. This analysis serves as a reference to study on the impulse-dependent wave transmission characteristic of the considered viscoelastic phononic crystal. The Sec. 5.4 introduces a new hybrid SHPB system (i.e., metal input bar and polymeric output bar) to resolve experimental challenges relating to low signal-to-noise ratios and input impulse magnitudes. As the applied impulse increases in the proposed hybrid SHPB apparatus, it is anticipated that there would be the appearance of some low transmission frequency zones, which are not identified from the linearly perturbed settings such as the analytical solution and the electrodynamic shaker tests.

### 5.2 Considered Viscoelastic Phononic Crystal and Loading Conditions

This section describes the details of mechanical properties of constituents adopted for a continuum phononic crystal (i.e., layered composite) under investigation. In addition, it introduces a brief overview of the two dynamic excitation conditions that are explored in this study to experimentally identify the wave transmission behavior of the considered layered composite.

#### 5.2.1 Material, Constitutive Model, and Specimen

The impulse-dependent wave transmission behavior of a bilayered composite consisting of metal and polymeric material is explored. For metal, we selected aluminum alloy 6061-T6, which has the mass density \( \rho_1 = 2700 \text{ kg/m}^3 \), the frequency-independent elastic moduli of Young’s modulus \( E_1 = 70.0 \text{ GPa} \), and Poisson’s ratio \( \nu_1 = 0.33 \). For polymeric material, silicone-based rubber
(Elite Double 32, Zhermack) is chosen, whose mass density and Poisson’s ratio are found to be \( \rho_2 = 1160 \, \text{kg/m}^3 \) and \( \nu_2 = 0.495 \) from measurements, respectively. This silicone rubber has been adopted by several other researchers \([11, 120]\), and the current investigation considers both the finite-deformation characteristics and viscoelastic properties of the silicon rubber.

Firstly, the finite-deformation characteristics of the silicone rubber are identified by performing quasi-static uniaxial compression tests at the strain rate of \( \dot{\varepsilon} = 1 \times 10^{-3}/\text{s} \), and the loading-unloading behavior is presented by the green solid line in Fig. 5.1A. Then, the equilibrium stress-strain path was determined by taking the mid-path of stress-strain curves from the uniaxial loading-unloading tests \([17, 172]\). The black dotted line in Fig. 5.1A shows that the constitutive behavior of the equilibrium path is accurately captured by a Yeoh hyperelastic model \([201]\), whose strain energy is:

\[
U = \sum_{j=1}^{3} \left[ C_{j0} \left( \bar{I}_1 - 3 \right)^j + \frac{1}{D_j} [J - 1]^{2j} \right],
\]

where \( C_{10} = 199.2 \, \text{kPa}, C_{20} = 16.43 \, \text{kPa}, C_{30} = 23.22 \, \text{kPa} \), and \( D_1 = 30.0 \, \text{GPa}^{-1} \), \( D_2 = D_3 = 0.0 \). Here, \( J = \det(F) \), \( \bar{I}_1 = J^{-2/3} \text{tr}(F^T F) \), and \( F \) is the deformation gradient. Secondly, the frequency-dependent viscoelastic properties of the silicone rubber were evaluated through dynamic mechanical analysis (DMA) experiments using RSA-G2 Solids Analyzer (TA Instruments) by applying infinitesimal strain. From the experiments, the complex-valued Young’s modulus \( \hat{E}_2(\omega) = \hat{E}_2'(\omega) + i \hat{E}_2''(\omega) \) is obtained, where \( \hat{E}_2' \) and \( \hat{E}_2'' \) denote the storage modulus and the loss modulus, respectively. Figure 5.1B shows the obtained frequency-dependent viscoelastic moduli within the sonic frequency range (here, up to 16 kHz). Note that the angular frequency \( \omega = 2\pi f \) is used in the text, but all the graphs are presented in the linear frequency \( f \). In addition, the generalized Maxwell model is adopted to represent the time-dependent behavior of the considered silicon rubber, whose relaxation modulus \( E_2(t) \) can be captured by the Prony series \([34]\):

\[
E_2(t) = E_{2,\infty} + \sum_{j=1}^{10} E_j e^{-\frac{t}{\tau_j}}
\]

where \( e \) is the Euler constant; \( E_{2,\infty} \) denotes the equilibrium (quasi-static) modulus; and each viscoelastic branch \( j \) is characterized by an elastic modulus \( E_j \) and a relaxation time \( \tau_j \). A rheological tool kit (IRIS \([194]\)) was adopted to identify the elastic equilibrium modulus \( E_{2,\infty} = 1.196 \, \text{MPa} \) and the Prony series coefficients \( E_j \) and \( \tau_j \) (see Tab. 5.1), and the black dotted
line in Fig. 5.1C shows the resulting relaxation modulus $E_2(t)$ at infinitesimal strains. In the
time-domain FE simulations presented in Sec. 5.3 and 5.4, we adopted the so-called finite-strain
viscoelastic model \[43, 115\], where the infinitesimal-strain equilibrium modulus $E_{2,\infty}$ in (5.2) is
replaced by the tangent modulus determined by the Yoeh hyperelastic model in (5.1). All the
numerical simulations in this study are performed using a commercial FE software, ABAQUS \[1\],
where the finite-strain viscoelastic model is implemented.

The considered bilayered composite specimen is composed of three unit-cells of aluminum
($d_1 = 5 \text{ mm}$) and silicone rubber layers ($d_2 = 20 \text{ mm}$). The cylindrical specimen had the
diameter of 45 mm and the overall height of 80 mm (see Fig. 5.2A), and the identical specimen is
employed for both shaker and SHPB tests in Sec. 5.3 and 5.4. Note that the specimen dimensions
(i.e., the thickness and the number of unit-cells) are guided by the analytical study in Sec. 5.3, so
that multiple low-transmission zones in the sonic frequency range can be experimentally observed,
which could be achieved by an electrodynamic shaker and SHPB apparatus in the lab.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
Branch Number $j$ & Modulus $E_j, [\text{MPa}]$ & Relaxation Time $\tau_j, [\text{s}]$ \\
\hline
1 & 0.477 & $4.99 \times 10^{-8}$ \\
2 & 0.328 & $4.83 \times 10^{-7}$ \\
3 & 0.221 & $3.55 \times 10^{-6}$ \\
4 & 0.157 & $1.72 \times 10^{-5}$ \\
5 & 0.113 & $6.04 \times 10^{-5}$ \\
6 & 0.127 & $2.28 \times 10^{-4}$ \\
7 & 0.114 & $1.33 \times 10^{-3}$ \\
8 & 0.077 & $9.07 \times 10^{-3}$ \\
9 & 0.059 & $6.42 \times 10^{-2}$ \\
10 & 0.043 & $4.89 \times 10^{-1}$ \\
\hline
\end{tabular}
\caption{Prony series coefficients of the considered silicone rubber}
\end{table}

### 5.2.2 Consideration of External Excitations

Assuming linear wave motion with infinitesimal deformation, the dispersion relation of phononic
crystals composed of dissipative medium is commonly characterized by two experimental ap-
proaches: driven-wave or free-wave conditions. While steady-state harmonic oscillations are
typically adopted for driven-wave conditions \[48, 59\], pulse-type loading (e.g., small force excita-
tion or small deformation) is exerted to phononic crystals for free-wave conditions \[177, 206\]. For
the conditions of infinitesimal deformation, the dispersion relations obtained from driven-wave
Fig. 5.1: Viscoelastic properties of the considered silicone rubber. (A) Loading-unloading stress-strain relation, whose equilibrium path is captured by the Yeoh model. (B) Frequency dependent storage modulus $\hat{E}'(\omega)$ and loss modulus $\hat{E}''(\omega)$. (C) Time dependent relaxation modulus.
and free-wave conditions are found to be similar for phononic crystals having low damping [7].

Before investigating impulse-dependent wave transmission behavior of viscoelastic phononic crystals, a reference study is conducted to establish amplitude-independent property. Specifically, a set of harmonic base excitation tests using an electrodynamic shaker is performed, which generates a voltage chirp signal in the sonic frequency range. The applied forces acting on the considered specimen were small, and consequently the corresponding deformations were infinitesimal. To lay out the approach, the experimental results from the harmonic base excitation tests are presented. Moreover, the corresponding time-domain FE analysis results in Sec. 5.3. Subsequently, the impulse-dependent wave transmission behavior of viscoelastic phononic crystals is investigated by using SHPB apparatus, which allows high impact excitation with nonlinear wave motion in the specimen. Under compressive impact conditions, the silicone rubber layers of the specimen are not in the linear range, showing up to the strain level of $\varepsilon_{tr} = -0.5$. Section 5.4 discusses the experimental results from impact excitation with SHPB and the corresponding time-domain FE simulation results.

### 5.3 Harmonic Excitation Analysis and Results

This section discusses the harmonic base excitation conditions inducing only small deformation in the considered layered composite specimen, and the results serve as a reference to the study on impulse-dependent wave transmission characteristics discussed in Sec. 5.4. The analytical dispersion relation of waves perpendicular to the layers in *infinitely periodic* viscoelastic layered
composites is reviewed, whose detailed derivations can be found in references [75, 126, 183]. Then, the details of experimental study using an electrodynamic shaker and the corresponding time-domain FE simulations are described.

5.3.1 Analytical Dispersion Relation and Transmission Coefficient

As described in Sec. 5.2.1, a bilayered composite consisting of alternating viscoelastic and elastic solids is considered. By solving the governing equation of motion together with Bloch periodic boundary conditions [8], two decoupled eigenvalue problems for dilatational and distortional waves can be simultaneously obtained. Here, only the dilatational wave motion is considered, since compressive waves are mainly considered in the experiments designed in this study. The subscript \( j \) is employed to refer the characteristics of the \( j \)-th layer (i.e., \( j = 1 \) for the aluminum layer and \( j = 2 \) for the silicone rubber layer). For instance, the periodic unit-cell length \( a \) is determined by \( a = d_1 + d_2 \), where \( d_j \) is the \( j \)-th layer thickness. Assuming steady-state harmonic wave motion in the composite, the displacement field \( \bar{u}_{j,n} \) and the normal stress field \( \bar{\sigma}_{j,n} \) of compressive plane waves for the \( j \)-th layer in the \( n \)-th unit cell [75] can be expressed as:

\[
\bar{u}_{j,n}(x_{j,n}) = P_{F,j,n} e^{-i \frac{\omega x_{j,n}}{c_{p,j}}} + P_{B,j,n} e^{i \frac{\omega x_{j,n}}{c_{p,j}}},
\]

\[
\bar{\sigma}_{j,n}(x_{j,n}) = -P_{F,j,n} \omega \rho_j c_{p,j} e^{-i \frac{\omega x_{j,n}}{c_{p,j}}} + P_{B,j,n} \omega \rho_j c_{p,j} e^{i \frac{\omega x_{j,n}}{c_{p,j}}},
\]  

(5.3)

where \( x_{j,n} \) represents the local \( x \)-coordinate for the \( j \)-th layer in \( n \)-th unit cell; \( P_{F,j,n} \) and \( P_{B,j,n} \) are the frequency-dependent complex displacement amplitude to be determined from boundary conditions; \( c_p(\omega) = \sqrt{\frac{E(\omega)[1-\nu]}{\rho[1+\nu][1-2\nu]}} \) is the frequency-dependent dilatational wave velocity. By applying the successive stress and displacement boundary conditions at the interfaces, the following transfer relation between the complex-valued displacements amplitude vector \( \mathbf{W} \) of adjacent unit-cells is obtained:

\[
\mathbf{W}_{1,n+1} = \mathbf{T} \mathbf{W}_{1,n}
\]  

(5.4)

where

\[
\mathbf{W}_{1,n} = \begin{bmatrix} P_{F,1,n} \\ P_{B,1,n} \end{bmatrix}, \quad \mathbf{W}_{1,n+1} = \begin{bmatrix} P_{F,1,n+1} \\ P_{B,1,n+1} \end{bmatrix},
\]  

(5.5)

\[
\mathbf{T} = R_1^{-1} R_2 D_2 R_2^{-1} R_1 D_1
\]  

(5.6)
\[ R_j = \begin{bmatrix} 1 & 1 \\ -\omega \rho_j c_{p,j} & \omega \rho_j c_{p,j} \end{bmatrix}, \quad D_j = \begin{bmatrix} e^{-i \omega d_j / c_{p,j}} & 0 \\ 0 & e^{i \omega d_j / c_{p,j}} \end{bmatrix}. \] (5.7)

Here, \( T \) is the frequency-dependent transfer matrix for dilatational wave motion perpendicular to the layers and determines the relation between the displacement amplitude vectors of adjacent unit-cells. In addition to the continuous stress/displacement boundary conditions (5.4), the Bloch periodic boundary conditions between adjacent unit-cells [8] can be employed as:

\[ W_{1,n+1} = e^{i \kappa_{pc} a} W_{1,n} \] (5.8)

where \( \kappa_{pc} = \kappa_{pc}^R + i \kappa_{pc}^I \) is the complex-valued wavenumber of the considered phononic crystals in the direction perpendicular to the layers. While \( \kappa_{pc}^R \) describes the phase of wave propagation, \( \kappa_{pc}^I \) is relating to the amplitude of wave attenuation. The application of Bloch periodic boundary conditions (5.8) to (5.4) provides an eigenvalue problem for dilatational wave motion:

\[ TW_{1,n} = e^{i \kappa_{pc} a} W_{1,n} \] (5.9)

By solving the above eigenvalue problem (5.9), the frequency-dependent dispersion relations of the considered bilayered composite for dilatational wave motion perpendicular to the layers [75, 126, 183] is obtained:

\[ \cos (\kappa_{pc} a) = \cos \left( \frac{\omega d_1}{c_{p,1}} \right) \cos \left( \frac{\omega d_2}{c_{p,2}} \right) - \frac{1}{2} \left( \rho_1 c_{p,1} + \rho_2 c_{p,2} \right) \sin \left( \frac{\omega d_1}{c_{p,1}} \right) \sin \left( \frac{\omega d_2}{c_{p,2}} \right). \] (5.10)

For the considered phononic crystals, the analytical dispersion relation (5.10) illustrates the characteristics of wave propagation \((\omega - \kappa_{pc}^R)\) and wave attenuation \((\omega - \kappa_{pc}^I)\), which are shown in Fig. 5.3A and 5.3B, respectively. Unlike in elastic phononic crystals, Fig. 5.3B illustrates the absence of \( \kappa_{pc}^I = 0 \) and \( d\omega / d\kappa_{pc}^R = \infty \), which indicates that waves in the considered viscoelastic-elastic phononic crystals possesses neither absolute band-gap nor absolute passing-band due to the frequency-dependent dilatational wave velocity of the silicon rubber \( c_{p,2}(\omega) \). In other words, waves in viscoelastic phononic crystals simultaneously propagate and attenuate at all frequencies.

Here, a wave transmission coefficient \( C_t(\omega) \) in introduced by taking the amplitude ratio between input force spectrum \( F_{\text{in}}(\omega) \) and output acceleration spectrum \( A_{\text{out}}(\omega) \):

\[ C_t(\omega) = \frac{\|A_{\text{out}}(\omega)\|}{\|F_{\text{in}}(\omega)\|}. \] (5.11)
where $\| \square \|$ denotes the Euclidean norm. Then, the wave transmission coefficient based on the analytical dispersion relation reads:

$$C_t(\omega) = \frac{\| \omega^2 W_{1,n+N} \|}{\| \omega \rho_1 c_{p,1} W_{1,n} \|} = \frac{\omega}{\rho_1 c_{p,1}} e^{-\kappa p c N a},$$  \hspace{1cm} (5.12)

where the denominator signifies the force vector acting on the aluminum layer in $n$-th unit-cell and the numerator represents the acceleration vector of aluminum layer in $(n + N)$-th unit-cell.

The analytical transmission coefficients for three different unit-cell spacings ($N = 1, 3, 5$) are shown in Fig. 5.3C, where the coefficient magnitude is presented in $dB$, i.e., $20 \log_{10} (C_t)$. As shown in Fig. 5.3C, the magnitude of the transmission coefficient $C_t$ is affected by $N$, but the frequency-zones indicating the low transmission are hardly affected by $N$. In the considered sonic frequency range, three low-transmission frequency zones are observed: first zone around 5 kHz, second zone around 10 kHz, and third zone around 16 kHz. As discussed in Sec. 5.2.1, $N = 3$ is chosen to design the specimens used in this study.

### 5.3.2 Base Excitation Tests with Electrodynamic Shaker

Electrodynamic shakers are commonly employed to conduct experiments for low amplitude dynamic excitation. By sweeping frequencies (i.e., chirp signal) through shakers, many researchers have investigated wave transmission characteristics of various periodic structures [41, 149, 197]. Figure 5.4A shows the harmonic excitation test set-up, where an electrodynamic shaker (B&K vibration exciter 4809) was anchored on an optical table. A force transducer (B&K 8230) was assembled to the shaker, and the layered composite specimen was vertically mounted on top of the force transducer. Then, an accelerometer (B&K 4394) was attached at top of the specimen. Swept-frequency chirp voltage signal (up to 16 kHz for 1 s) produced by a waveform generator (NI PXI-5412 with B&K amplifier 2718) was sent to the electrodynamic shaker. Both the force and acceleration time-history measurements were collected at a sampling rate of 1 MHz by a data acquisition system (NI PXI-5105 in NI PXI-1042Q).

The electrodynamic shaker tests are conducted for two different maximum force levels: one with 21.0 N and the other with 53.7 N. By taking the fast Fourier transform (FFT) of the input force acting on the specimen bottom and the output acceleration on the specimen top, the wave transmission coefficient $C_t(\omega)$ is calculated using (5.11). Fig. 5.5A-1 and 5.5A-2 show the experimental transmission coefficient spectra $C_t(\omega)$ for the maximum force levels of 21.0 N and
Fig. 5.3: Pressure wave characteristics of the infinitely periodic viscoelastic-elastic phononic crystal under consideration. (B) Phase dispersion relation, $\kappa_{pc}^R - f$. (B) Attenuation relation, $\kappa_{pc}^I - f$. (C) Transmission coefficient obtained from (5.11), $C_t$. 
53.7 N, respectively. Note that the force level of 53.7 N is obtained by applying a voltage level close to the specification limit of the electrodynamic shaker. The solid lines in Fig. 5.5A represent the average of five tests for each force level, and the shaded area along the solid line shows the corresponding standard deviation. From Fig. 5.5A, three low transmission frequency zones are found: first zone around 2 kHz, second zone around 10 kHz, and third zone around 14 kHz. Although the electrodynamic shaker tests consider the finite-size layered composite specimen under transient harmonic excitations, the experimentally obtained low transmission zones are found to be in the vicinity of the analytical predictions based on infinitely periodic phononic crystals under steady-state harmonic excitations. More importantly, the comparison between Fig. 5.5A-1 and 5.5A-2 confirms that the experimentally-obtained transmission coefficient spectra are nearly independent of the amplitude of excitation forces generated by the electrodynamic shaker.

### 5.3.3 Time-domain FE Simulations

For the base excitation tests with the electrodynamic shaker, the time-domain simulations using the commercial FE code ABAQUS/Standard are also conducted. All the numerical simulations are performed using a 2-D axisymmetric model with 4-node bilinear element CAX4R. A mesh refinement study confirmed that the mesh sweeping size of 1.25 mm (see Fig. 5.2B) is sufficiently small to obtain the convergence of the FE simulation results. In order to minimize the overall drift of the specimen displacement, an equivalent base acceleration is applied with the baseline correction procedure [21, 111], instead of using the direct force time-history measured from the experiments. The equivalent base accelerations are obtained from the experimentally measured bottom force histories divided by the specimen mass. The acceleration histories on the FE model top were collected to calculate the transmission coefficient $C_t(\omega)$.

Figure 5.5B-1 and 5.5B-2 show the numerically-obtained transmission coefficient spectra $C_t(\omega)$ for the maximum force levels 21.0 N and 53.7 N, respectively. Manifesting the three low transmission frequency zones around 2 kHz, 10 kHz, and 14 kHz, numerical simulations are in good agreement with their experimental counterpart. From the outcomes of both experiments and simulations, the impulse-independent transmission coefficient spectra of the considered composite specimen can be obtained by applying harmonic base excitations through a electrodynamic shaker inducing small forces.
Fig. 5.4: (A) Electrodynamik shaker test set-up. (B) SHPB test set-up.
Fig. 5.5: Amplitude-independent transmission characteristics. (A) Transmission coefficient $C_t(\omega)$ obtained from the base excitation tests with electrodynamic shaker: (A-1) 21.0 N, (A-2) 53.7 N. Note that the shaded area denotes the standard deviation. (B) The corresponding results from the time-domain FE simulations: (B-1) 21.0 N, (B-2) 53.7 N.
5.4 Impact Excitation Analysis and Results

This section investigates the impulse-dependent wave transmission behavior of the considered viscoelastic phononic crystal specimen by exerting high impact excitation through SHPB apparatus. Typically, SHPB apparatus is known as a standard set-up for high strain-rate tests (e.g., up to the strain-rate of 5000/s or higher) [170], and it is composed of three bars: striker, input bar, and output bar (Fig. 5.4B). A specimen is placed between the input and output bars. By shooting the striker using a gas gun, a stress wave is created and pass through the bars and the specimen. The stress-strain response of the specimen can be calculated from the travelling waves in the bars using a simple 1-D wave propagation theory in elastic medium [170]. Recently, the idea of using SHPB apparatus for phononic band-structure study is introduced by Feng and Liu [60, 61], who have reported the stress-induced band-gap tunability of polymer-metal phononic crystals. This section first discusses experimental issues in regard to the application of SHPB for viscoelastic phononic crystals, and then propose a hybrid SHPB apparatus to overcome the issues.

5.4.1 Considerations on SHPB Tests for Viscoelastic Phononic Crystals

Feng and Liu [60, 61] have investigated 1-D phononic crystals made of steel/epoxy and aluminum/epoxy using metallic SHPB apparatus, whose input and output bars were solid steel rods. Assuming a linearly perturbed setting, they compared experimentally-observed phononic band-structures with numerical results obtained from elastic FE models. Note that their specimens included a viscoelastic material, epoxy [140], whose Young’s modulus (i.e., $E_{\text{epoxy}} \approx 4 \text{ GPa}$) is nearly two orders of magnitude smaller than that of the apparatus bar material (i.e., steel having $E_{\text{steel}} \approx 210 \text{ GPa}$).

The viscoelastic property of polymeric specimens and the impedance mismatch between polymeric specimens and metallic apparatus bars are critical to correctly analyze the experimental results from SHPB apparatus. In particular, several researchers have discussed solutions to address this impedance mismatch by using either polymeric bars [65, 188] or hollow metallic bars [44, 147]. Simply speaking, the impact energy to linearly deform the polymeric specimens is not sufficiently large to provide high signal-to-noise ratios on solid metallic apparatus bars. On the other hand, the impact energy which is sufficient large to provide the desired high signal-to-noise
ratios on solid metallic apparatus bars induces finite viscoelastic deformations in the polymeric specimens. Moreover, the frequency spectrum of the incident wave on the input bar is typically characterized by amplitude-drops at some frequencies, and these inherent amplitude-drops of the incident wave spectrum are primarily dependent on the striker length \([4, 141]\). Oversight of this limitation may result in the erroneous identification of phononic band-structure near those amplitude-drop frequencies. Thus, when SHPB apparatus is adopted for viscoelastic phononic crystal study, these critical points should be thoroughly addressed in order to properly identify phononic band-structures, i.e., wave transmission behavior.

5.4.2 Hybrid SHPB Configuration

Since high impact energy should be applied to specimens, metallic SHPB apparatus can be adopted for producing high signal-to-noise ratios in the incident wave. However, due to the material damping of viscoelastic layers (silicon rubber), the transmitted wave strain signal on the output bar is significantly weak, causing low signal-to-noise ratios. On the other hand, polymeric SHPB apparatus could be considered to improve signal-to-noise ratios, but the force generated within the polymeric bars is insufficient to create finite deformation within the specimens while maintaining apparatus bars in their linear range to ensure 1-D elastic wave motions \([13, 103]\). In other words, in order to investigate the impulse-dependent wave transmission behavior of viscoelastic phononic crystals, a suitable SHPB configuration is required that simultaneously allows large input force to specimens and captures weak transmitted waves on the output bar. In order to overcome this challenge, a hybrid SHPB system is developed that consists of a metallic input bar and a polymer output bar. Especially, an aluminum input bar (type: 6061-T6, length: 3 \(m\), diameter: 45 \(mm\)) and a nylon output bar (type: PA-66, length: 3 \(m\), diameter 50 \(mm\)) with aluminum strikers of various lengths are configured for experimental study. Strain gauges (EA-13-031CF-120/E from Vishay Measurements Group) are placed on each apparatus bar, and the strain signals were amplified by a signal amplifier (Vishay 2310). Then, the amplified strain signals are recorded using a DAQ system (NI PCI-6115) at a sampling rate of 2.5 \(MHz\). In addition, the launching speed of striker on impact was recorded by a photoelectric sensor (Tri-tronics XP10). Additional details of the hybrid SHPB apparatus are given in Tab. 5.2.

The measured strain waves on SHPB bars need to be transported to the specimen-bar interfaces to obtain the force and the acceleration acting on the considered specimen. The geometric and material properties of the apparatus bars affect wave propagations within the bars,
whose wave characteristics can be captured by their dispersion relation. The complex-valued dispersion relation, $\kappa_{\text{bar}}(\omega) = \kappa^R_{\text{bar}}(\omega) + i \kappa^I_{\text{bar}}(\omega)$, can be experimentally obtained by performing the single-bar impact test [13, 170]. Note that $\kappa^R_{\text{bar}}$ and $\kappa^I_{\text{bar}}$ represent the frequency-dependent propagation and attenuation characteristics of waves in the bar, respectively. The single-bar impact tests are conducted for the considered aluminum and nylon bars, and then the phase velocity $c_{\text{bar}}(\omega) = \omega/\kappa^R_{\text{bar}}(\omega)$ and the attenuation coefficient $\kappa^I_{\text{bar}}(\omega)$ are experimentally obtained, which are presented by the solid lines for the average and the shade areas for the standard deviation in Fig. 5.6. Note that the red dashed lines in Fig. 5.6A-1/B-1 also show the Pochhammer-Chree solution [67] for the corresponding elastic counterpart. These experimentally-obtained coefficients were used throughout our study with the hybrid SHPB apparatus.

For the investigation of wave transmission behavior of specimens using SHPB, ideal incident waves would be sharp impulse time-signals, which resemble the dirac-delta function containing infinite frequency contents. Furthermore, the impulse of ideal incident waves should be high enough to excite the impulse-dependent wave characteristics of specimens. In practice, a sharp impulse signal having a short impulse duration can be archived by adopting a short striker. However, it requires very high launching speed to produce desirable amount of impulse due to its small mass, but the striker launching speed is typically limited by the specification of a given SHPB striker-launching system. On the other hand, a long striker having a large mass can generate high impulse with low launching speeds, but it suffers frequency amplitude-drops in the incident wave spectrum [4, 71, 193]. An amplitude-drop represents the scarcity of some frequency contents in an incident wave. Consequently, regardless of wave characteristics of the considered specimen, the transmitted signal may lack some frequency contents near the amplitude-drops, entailing a distortion of transmission coefficients. Thus, prior to the main SHPB impact tests to be discussed in Sec. 5.4.3, we conducted a series of impact tests by shooting three different

---

**Tab. 5.2: Specifications of the hybrid SHPB apparatus.**

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Aluminum striker</th>
<th>Aluminum input bar</th>
<th>Nylon output bar</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length, $L$ [m]</td>
<td>0.05, 0.15, 0.25, 0.30</td>
<td>3.00</td>
<td>3.00</td>
</tr>
<tr>
<td>Diameter, $D$ [mm]</td>
<td>45</td>
<td>45</td>
<td>50</td>
</tr>
<tr>
<td>Longitudinal wave speed, $c_0(\omega = 0)$ [m/s]</td>
<td>5070</td>
<td>5070</td>
<td>1700</td>
</tr>
<tr>
<td>Mass density, $\rho$ [kg/m$^3$]</td>
<td>2700</td>
<td>2700</td>
<td>1140</td>
</tr>
<tr>
<td>Distance between strain gage and specimen-bar interface, $\Delta$ [m]</td>
<td>-</td>
<td>1.5</td>
<td>0.30</td>
</tr>
</tbody>
</table>
5. Amplitude-Dependent Wave in Periodic Viscoelastic Composite

Fig. 5.6: Wave propagation characteristics of SHPB apparatus bars. (A) Aluminum bar: (A-1) phase velocity $c_{al}(\omega)$, (A-2) attenuation coefficient $\kappa_{al}(\omega)$. (B) Nylon bar: (B-1) phase velocity $c_{ny}(\omega)$, (B-2) attenuation coefficient $\kappa_{ny}(\omega)$. Note that the average experimental results are denoted by the solid lines and the standard deviations are represented by the shaded areas. In addition, the dashed and the dotted lines indicate Pochhammer-Chree analytical solution and the numerically-obtained propagation coefficient from FE simulations, respectively.

Strikers ($L_{str} = 50, 150, 250 \text{ mm}$) to the input bar and collected incident strain waves on the input bar. By taking FFT of the incident strain signals, the frequency amplitude-drops are analyzed for each striker as shown in Fig. 5.7. The longer the striker is, the earlier the frequency amplitude-drop occurs in the frequency domain. Thus, the valid frequency limit of each striker is identified for the wave transmission coefficients of the considered specimen, i.e., up to around 20 $kHz$, 14 $kHz$, 8 $kHz$ for $L_{str} = 50, 150, 250 \text{ mm}$, respectively.

5.4.3 Impact Excitation Tests with Hybrid SHPB

The impact excitation tests with the SHPB apparatus are conducted on the viscoelastic phononic crystal specimen under three different striker loading conditions: (a) 50 mm-long striker with a launching speed of 12.18 m/s, (b) 150 mm-long striker with 10.04 m/s, and (c) 250 mm-long
5. Amplitude-Dependent Wave in Periodic Viscoelastic Composite

Fig. 5.7: Spectrum of input strain signals experimentally obtained from different striker lengths.

striker with 8.42 m/s. After taking the FFT of the incident and transmitted strain signals, the input force spectrum \( F(\omega) \) acting on the specimen and the output acceleration spectrum \( A(\omega) \) at the interface between the specimen and the output [170] are calculated using:

\[
F(\omega) = \frac{s_{al}\rho_{al}\omega^2}{\kappa_{al}(\omega)^2} \hat{\varepsilon}_{inc}(\omega) e^{i\kappa_{al}(\omega)\Delta_{al}},
\]

\[
A(\omega) = \frac{-i\omega^2}{\kappa_{ny}(\omega)} \hat{\varepsilon}_{tra}(\omega) e^{i\kappa_{ny}(\omega)\Delta_{ny}},
\]

where \( \hat{\varepsilon}_{inc}(\omega) \) and \( \hat{\varepsilon}_{tra}(\omega) \) represent the FFT of the incident and transmitted time signals measured on the apparatus bars, respectively; the subscripts \( al \) and \( ny \) represent aluminum input bar and nylon output bar, respectively; \( s \) and \( \rho \) denote the cross-sectional area and the density of the apparatus bar, respectively; \( \Delta \) is the distance between the location of the strain gauge on the apparatus bar and the specimen/bar interface. Then, using (5.11), the wave transmission coefficient \( C_t(\omega) \) are calculated which are summarized in Fig. 5.8A. Note that the dotted lines indicate the low-fidelity experimental results following the discussion regarding the effect of the striker length on incident waves in Sec. 5.4.2. Furthermore, by taking the inverse FFT of \( F(\omega) \), we re-constructed the time-history of the input force acting on the specimen. Then, the impulse exerted to the specimen was determined by integrating the force time-history over the loading duration: (a) the impulse of 5.60 N·s for the 50 mm-long striker case, (b) 13.2 N·s for the 150 mm-long one, and (c) 18.6 N·s for the 250 mm-long one. By focusing on the solid line in Fig. 5.8A-1, two low transmission frequency zones around 2 kHz and 8 kHz are found, which are close to the outcome of the electrodynamic shaker tests. Due to the low signal-to-noise ratio
depicted by the large standard deviation in Fig. 5.8A-1, the third low transmission frequency zone around 14 kHz could not be properly identified. Interestingly, however, it is found that high impulse produces a new low transmission frequency zone around 5.5 kHz (vertically shaded in Fig. 5.8A-3/4), which are neither predicted from the analytical solution from Sec. 5.3.1 nor observed from the electrodynamic shaker tests in Sec. 5.3.2.

5.4.4 Time-domain FE Simulations

For the impact excitation tests with the SHPB apparatus, the corresponding time-domain simulations are also conducted using ABAQUS/Explicit by modeling all the SHPB components (i.e., striker, input bar, output bar) as well as the specimen. All the numerical simulations were performed using a 2-D axisymmetric model with 4-node bilinear element CAX4R. Mesh sweeping sizes of 1.25 mm and 2.5 mm were adopted for the specimen and the bars, respectively. While a linear elastic material model is used for the aluminum input bar, a linear viscoelastic material model is applied for the nylon output bar. Considering the equilibrium (quasi-static) modulus $E_{ny,\infty} = 2870 \, MPa$ of the nylon bar material, the scaled Prony series parameters provided by Fujikawa and Takashi [62] is scaled to fit the current nylon material. After conducting an FE simulation for the single nylon bar impact test with this linear viscoelastic model, it is confirmed that the numerically-obtained phase velocity and attenuation coefficient (see black dotted lines in Fig. 5.6B-1/2) compare well with the experimentally measured quantities. Similar to the harmonic excitation simulation in Sec. 5.3.3, the silicon rubber is modeled using the finite viscoelastic model described in Sec. 5.2.1. In order to consider the friction between the specimen and the bars, FE simulations adopted the Coulomb friction model with friction coefficients of $\mu = 1.2$ [139] and 0.1 [86] for aluminum/aluminum and aluminum/nylon interfaces, respectively. For each test with a different striker length (see Sec. 5.4.3), the measured launching striker speed was applied as the initial boundary condition.

Figure 5.8B shows the numerically obtained transmission coefficient spectra $C_t(\omega)$ for all the corresponding impact excitation tests described in Sec. 5.4.3. Note that the solid lines indicate the reliable experimental results following the discussion regarding the effect of the striker length on incident waves in Sec. 5.4.2. In Fig. 5.8B-1, three distinctively low transmission frequency zones around 2 kHz, 8 kHz, and 14 kHz are observed. Note that the third low transmission frequency zone around 14 kHz is in the vicinity of the analytical prediction (16 kHz) based on infinitely periodic layered composites, but is not properly identified from the experiment due
Fig. 5.8: Amplitude-dependent transmission characteristics obtained from SHPB setting. (A) Transmission coefficient $C_t(\omega)$ from the SHPB tests: (A-1) 50 mm-long striker with the impulse of 5.6 N·s, (A-2) 150 mm-long striker with the impulse of 13.2 N·s, and (A-3) 250 mm-long striker with the impulse of 18.6 N·s. (B) The corresponding results from the time-domain FE simulations: (B-1) 50 mm-long striker with the impulse of 5.5 N·s, (B-2) 150 mm-long striker with the impulse of 13.6 N·s, and (B-3) 250 mm-long striker with the impulse of 18.5 N·s. (C) The numerical results obtained from the additional time-domain FE simulations where a 150mm-long striker is launched for all three cases: (C-1) the impulse of 5.5 N·s, (C-2) the impulse of 13.6 N·s, and (C-3) the impulse of 18.7 N·s.
to the low signal-to-noise ratio. The numerical simulation results presented in Fig. 5.8B also evidently show the emergence of a new impulse-dependent transmission frequency zone around 5.5 kHz that are observed in the experiments (Fig. 5.8A-3/4). In addition, Fig. 5.8B-1/2 show that there is another low transmission frequency zone around 12 kHz.

5.5 Discussion

This section first discusses the impulse-independent wave transmission behavior obtained from the harmonic excitation conditions. Secondly, the experimental and numerical results from impact excitation conditions are examined and the impulse-dependent wave transmission behavior of the considered viscoelastic phononic crystals is highlighted.

5.5.1 Impulse-independent Wave Transmission Behavior from Harmonic Excitation Conditions

The analytical wave transmission coefficient spectrum in Fig. 5.3C is obtained by assuming steady-state harmonic wave motions in infinitely periodic viscoelastic phononic crystals. For the electrodynamic shaker tests and the corresponding FE simulations, a finite size specimen is considered under the swept-frequency chirp signals. Despite these differences in the specimen and loading conditions, both the test and the simulation results exhibit three low-transmission frequency zones (i.e., one around 2 kHz, another near 10 kHz, and the last one around 14 kHz shown in Fig. 5.5), which are placed in the vicinity of the analytical predictions. The electrodynamic shaker tests and the corresponding FE simulations show that these three low transmission frequency zones are found to be independent of the harmonic base excitation force levels applied by the electrodynamic shaker.

In order to better understand the results of the electrodynamic shaker tests compared to those of the SHPB impact tests, the study also considers the energy-equivalent conversion of a swept-frequency chirp signal into a pulse signal. Consider a unit-amplitude swept-frequency chirp signal having a time duration $T_c$, a linear frequency bandwidth $B_c$, a starting frequency $f_c$, and the linear rate of frequency change $k_c = B_c/T_c$. For this unit-amplitude chirp signal, Cook and Klauder [49, 87] provide a closed-form expression for an artificial energy-equivalent pulse...
signal \( p_c(t) \):

\[
p_c(t) = \left| \sqrt{B_c T_c} \frac{\sin(\pi B_c t)}{\pi B_c t} e^{i \pi (f_c t - k_c t^2/2)} \right| ,
\]

whose central peak pulse has the amplitude of \( \sqrt{B_c T_c} \) and the time duration of \( 2/B_c \). Note that the converted pulse signal in (5.14) is related to an exponentially-decaying sinc function in time domain, and the energy stored in the signal mainly resides within the central peak pulse \([87]\). Thus, the energy-equivalent pulse force time-signal is obtained by multiplying (5.14) with the average of each swept-frequency chirp force signal, and then the impulse is calculated by taking the time-integration. Based on this procedure, the base excitation force levels of 21.0 \( N \) and 58.7 \( N \) are found which correspond to the impulse level of 0.04 \( N \cdot s \) and 0.12 \( N \cdot s \), respectively. Thus, the energy-equivalent conversion of swept-frequency chirp signals reveals that the converted pulses from the electrodynamic tests are approximately two orders of magnitude smaller than the applied impulses in the SHPB impact tests.

### 5.5.2 Impulse-dependent Wave Transmission Behavior from Impact Excitation Conditions

To examine the effectiveness of the hybrid SHPB apparatus for the impulse-dependent transmission behavior study, three different levels of impulse were explored by changing the striker length and its launching speed as presented in Sec. 5.4.3. At the impulse level of 5.6 \( N \cdot s \), both the SHPB tests and the corresponding FE simulation (see Fig. 5.8A-1/B-1) show two distinctively low transmission frequency zones around 2 \( kH z \) and 8 \( kH z \), which are close to the harmonic base excitation results. However, as the applied impulse increases, Fig. 5.8A/B show the appearance of a new low transmission frequency zone around 5.5 \( kH z \) and 12 \( kH z \), which are not observed from the harmonic base excitation conditions. Recall that the reliable frequency ranges from SHPB tests are adversely affected by the striker length.

In order to assure the appearance of impulse-dependent low transmission frequency zone of the considered viscoelastic phononic crystal, a set of additional FE simulations is further conducted, which are free from the limitations relating to signal-to-noise ratios and striker launching speeds. In the additional FE simulations, the 150 \( mm \)-long aluminum striker is selected, which provides the valid frequency limit up to 14 \( kH z \) based on the single-bar impact test discussed in Sec. 5.4.2. In order to achieve high impulse, the launching speed is increased instead of using longer strikers. Striker launching speed in the FE simulations are determined to provide similar impulse
magnitudes corresponding to the cases where various striker lengths are explored in Fig. 5.8B. For example, 150 mm-long striker with the launching speed of 8 m/s produces the impulse of 5.5 N·s, which is close to 5.6 N·s generated by using 50 mm-long striker with the launching speed of 12.18 m/s. The results of the additional FE simulations are summarized in Fig. 5.8C, where the graphs in the same row have a similar magnitude of impulse acting on the specimen. By comparing the graphs in Fig. 5.8B/C side-by-side, the location of low transmission frequency zones in Fig. 5.8C are well compared with ones in Fig. 5.8B within the reliable frequency limits despite the striker length difference. This good agreement suggests that impulse serves as a proper measure to investigate the nonlinear wave transmission behavior of the considered viscoelastic phononic crystals. As the applied impulse increases, Fig. 5.8B clearly shows the appearance of the impulse-dependent low transmission frequency zones near 5.5 kHz and 12 kHz. In particular, note that a second impulse-dependent low transmission frequency zone around 12 kHz cannot be identified from the 300 mm-long striker.

In order to further investigate the qualitative change in transmission spectra affected by the applied impulse, an additional series of FE simulations are performed by sweeping various striker launching velocities. Figure 5.9A shows a contour plot of transmission coefficient $C_t(\omega)$ of the considered viscoelastic phononic crystal specimen. The contour plot clearly demonstrates the evolution of transmission coefficient with respect to the applied impulse. At low impulse (e.g., 5 N·s), there are two low transmission frequency zones around 2 kHz and 8 kHz (dark brown color). However, new low transmission frequency zones near 5.5 kHz and 12 kHz emerge around the impulse of magnitude 15 N·s and 18 N·s, respectively. Practically, the nonlinear wave transmission behavior can also be evaluated by using the ratio between the input force and the output force acting on the phononic crystals. So, the study introduces an additional transmission coefficient spectra $\tilde{C}_t(\omega) = \|F_{out}(\omega)\|/\|F_{in}(\omega)\|$, where $F_{out}(\omega)$ and $F_{in}(\omega)$ denote the output and input force spectrum, respectively. A contour plot of force transmission coefficient $\tilde{C}_t(\omega)$ for the considered viscoelastic phononic crystal is presented in Fig. 5.9B. Similar to the results shown in Fig. 5.9A, the contour plot of $\tilde{C}_t(\omega)$ in Fig. 5.9A also distinctively illustrates the emergence of low transmission frequency zones at 5.5 kHz and 12 kHz as the applied impulse increases. Together with the SHPB experiments shown in Fig. 5.8A, these additional sets of simulation results demonstrate that the impulse-dependent wave transmission behavior can be experimentally investigated by adopting the hybrid SHPB apparatus.
5.6 Conclusions

Viscoelastic polymers are commonly employed together with metals as a constituent of phononic crystals to exploit high impedance mismatch. Recently, there has been a rising interest in the nonlinear wave transmission characteristics of phononic crystals. However, due to the inherent damping properties of viscoelastic polymers, conventional electrodynamic shakers and piezoelectric actuators are not suitable to generate sufficiently large excitation to induce nonlinear wave motion in viscoelastic phononic crystals. Thus, experimental studies have been predominantly limited to the 1-D chain of beads under impact exerted in customized impact apparatus. This study proposes a hybrid SHPB system and examines it as a tool to study the impulse-dependent wave characteristics of 1-D continuum viscoelastic phononic crystals. The proposed hybrid SHPB apparatus comprises an aluminum input bar and a nylon output bar in order to resolve experimental challenges related to signal-to-noise ratios and input impulse magnitudes. While the aluminum input bar allows high forces acting on the specimen, the nylon output bar improves the signal-to-noise ratio in the transmitted signals. Using the hybrid SHPB apparatus, several low transmission frequency zones are observed, which are not identified from the linearly perturbed settings such as the analytical solution and the electrodynamic shaker tests. Furthermore by conducting a series of additional FE simulations, the study ensures the appearance of impulse-dependent wave transmission characteristics. (A) $C_t(\omega) = \|A_{out}(\omega)\|/\|F_{in}(\omega)\|$ defined in (5.11). (B) $\tilde{C}_t(\omega) = \|F_{out}(\omega)\|/\|F_{in}(\omega)\|$. Note that the dark brown color indicates the low transmission frequency zones.
dependent low transmission frequency zones of the considered viscoelastic phononic crystal specimen. The additional sets of simulations further illustrate the impulse-dependent evolution of transmission coefficient, and they demonstrate that the impulse-dependent wave transmission behavior can be experimentally investigated by adopting the hybrid SHPB apparatus.

SHPB apparatus is widely used as a standard set-up for high strain-rate tests of materials, and this study proposes a novel utilization such that it can also be properly used for nonlinear wave propagation characterisation of viscoelastic phononic crystals. The current study particularly investigates the compression SHPB setup to evaluate wave transmission characteristics, but the experimental procedure and analysis presented in this study can also be adopted to other SHPB setups such as tension [66, 119], torsion [57, 196], and shear [185] loading conditions. Thus, this work opens a new avenue to conventional SHPB apparatus, which can be employed to study the emerging research field of nonlinear wave characteristics of phononic crystals.
6. CONCLUSION

6.1 Summary

The current thesis is performed to characterize the elastodynamic wave motion in periodic layered composites. Specifically, the impending issues in analytical, numerical and experimental fields of periodic composite are investigated.

Due to lack of a proper analytical solution of sagittal plane wave motion in arbitrary direction of viscoelastic IPMC, the researches commonly inspect the dispersion relation for wave perpendicular to the layers. However, the dispersion relation can significantly deviate even for minor change in wave propagation angle which should be inspected for some applications. The complexity in analytical formulation has impeded a closed-form solution of the problem. Specifically, coupled P- and SV-waves requires mixed mode solution when wave propagation is oblique to the layers. Despite the effect of propagation angle on wave motion, the sagittal plane wave dispersion relation in viscoelastic IPMC is completely overlooked. Several analytical and experimental studies for wave propagation perpendicular to the viscoelastic composite have shown the importance of considering the damping behavior. Therefore, this thesis investigates the sagittal plane wave viscoelastic IPMC by considering the damping in analytical formulation of transfer matrix to obtain dispersion relation in the framework of complex-valued wavevector. In addition, this solution is also modified for periodic viscoelastic-elastic composite by incorporating an intricate wave transmission behavior of the layer interfaces. The analytical solution is evaluated for a practical composite consisting of aluminum and elastomer. Moreover, the influence of wave propagation angles on transmission behavior of composite is evaluated to illustrate the adequacy of the solution.

This study has also paid attention to the numerical dispersion relation of IPMC. Specifically, the well known problem of fictitious modes of numerical dispersion relation is resolved. Especially, the FE method which is commonly employed to incorporate the effect of material and geometric
nonlinearity, is highly susceptible to spectral distortion due to the use of 2-D elements for modeling periodic composite. In fact, the spectral distortion stems from artificial periodicity applied to the direction parallel to the composite layers which causes spatial aliasing in the wavevector domain. A systematic investigation is performed on fictitious modes for two influencing factors, e.g., unit cell aspect ratio and propagation angle. The sagittal plane wave solution is employed to regenerate the numerical fictitious modes based on the concept of spatial aliasing. Furthermore, the thesis also provides a guideline for avoiding spectral distortion by combining anti-aliasing condition and effective modulus theory. This comprehensive guideline is inspected for various aspect ratios of FE model and wave propagation angles to obtain accurate dispersion relation for required frequency range.

Most importantly, this study explores experimental aspects of viscoelastic composite to invigorate their practical applications. Due to the inadequacy of conventional experimental systems, the use of these crystals are limited within small vibration conditions. The present study overcomes this obstacle by developing a detailed procedure to experimentally determine amplitude-dependent wave transmission of continuum periodic crystal, which is further enhanced by corresponding numerical analysis. By employing a 1-D periodic viscoelastic-elastic composite, the transmission spectra is characterized for both small and large amplitude excitation using vibration shaker and hybrid SHPB system, respectively. Experimental evolution of transmission spectra is obtained as a function of impulse which represents the total energy of impact force. Outcome of the study shows development new low transmission zones at large impulse values which signifies the influence of wave amplitude on transmission property of the composite.

### 6.2 Original Contribution

In broader aspect, contribution of this research will enrich various elastodynamic wave propagation investigations of phononic crystals.

- The development of closed-form dispersion relations for viscoelastic IPMCs will enhance the analytical investigation of this area. Moreover, the theoretical evidence of viscoelasticity effect on dispersion relation will encourage the researches to cautiously consider the actual damping behavior of rubber materials. In addition, the complex valued dispersion relation elastic and viscoelastic IPMCs are efficiently used to characterize the wave transmission for arbitrary direction of propagation. This approach can be also followed for 2-D and 3-D
6. Conclusion

formation of crystals.

- The thorough explanation of spatial aliasing in numerical dispersion relation informs the community about potential appearance of fictitious modes. Because, the spectral distortion of dispersion relation is not limited to 1-D periodic composite, it can occur in any system where an artificial periodicity is used. For instance, the dispersion analysis has been performed by employing artificial periodicity for homogeneous medium [10, 18, 70, 72, 91, 131, 186] and out-of-plane direction of 2-D crystals [18, 19, 96]. Since, many researchers are unaware of the effect of spatial aliasing, the unit cells for numerical analysis are selected arbitrarily. Therefore, the resulting dispersion relations are susceptible to spatial aliasing. The outcome of current study will enable the researchers to avoid fictitious mode in numerical dispersion relations.

- The final contribution of the thesis establishes an experimental methodology for periodic crystals to characterize the amplitude-dependent wave motion. The efficiency and limitations of the SHPB test explained here can be followed as a guideline for any impact based excitation of phononic crystals. Furthermore, the impulse based evaluation of transmission property can be generalized for characterizing the effect of wave amplitude in 2-D and 3-D periodic systems. In addition, the concept of hybrid metal-polymer SHPB system can be adopted for general investigations of soft materials.

6.3 Future Investigations

Based on the outcome of the current dissertation, the following investigations can be perused in future:

- Spatial aliasing in dispersion relation of homogeneous solids and 2-D crystals: The frequency-dependent wave dispersion behavior is investigated for many modern materials. Dispersion analysis for the homogeneous materials are performed using arbitrary unit cells in FE-framework where Bloch periodic condition is employed in all directions. In addition, out-of-plane dispersion relation of 2-D phononic crystals are often investigated to characterize wave motion in that direction. Inevitably, a periodic boundary condition is imposed on the nonperiodic direction of the crystal. Due to application of artificial periodicity, the dispersion relations of these system are susceptible to spatial aliasing. Therefore,
investigations are required to identify fictitious modes and obtain necessary guidelines to avoid spatial aliasing for these systems.

• **Omnidirectional band-gap in elastic IPMC**: Generally, vibration and impact loads are localized at a single source. Under such conditions, the wave in periodic composites propagates as a spherical wave instead of plane wave. In fact, wave transmission property of the periodic composites vary as a function of propagation angle from the source. Consequently, wave transmitted on the output side doesn’t possess any definite band-gap. By combining a load distributing structure with periodic composites, the original band-gap for wave propagation perpendicular to the layers can be recovered. Since the conventional omnidirectional mechanism can only provide a narrow band-gap, the new approach should studied to retain the complete band-gap spectra.

• **Amplitude-Dependent wave in 2-D and 3-D phononic crystals**: The investigations on the effect of wave amplitude in 2-D and 3-D phononic crystals are limited. The study of these crystals should consider the complete Brillouin zones (BZ) which are relatively intricate. The wavevectors in different directions of BZ requires investigation for amplitude dependency. Furthermore, attention should be given to the effect of wave amplitude on complete band-gaps of 2-D and 3-D phononic crystals.

• **Increase usable frequency range of SHPB**: All SHPB systems (metallic, polymeric and hydraulic) are characterized by zero-crossings in FFT amplitude spectra. Such drop in spectral content can lead to misleading results. In order to increase the usable frequency range, the duration of the incident pulse should be reduced. In fact, the pulse duration of experimentally obtained signal can be significantly larger than the analytical value. However, the reason for this departure remains unexplored. Therefore, investigation should be conducted to increase the usable frequency range by reducing the pulse duration.

• **Effect of wave amplitude on geometric instability**: By employing geometric nonlinearity, the configuration of 2-D and 3-D phononic crystals can be significantly altered. At deformed shapes, the wave dispersion behavior of these crystals changes due to new formation of unit cells. This novel technique can be adopted to exploit the same crystal for multipurpose applications. For small amplitude vibration, the effect of geometric nonlinearity has been studied to obtain tunable band-gap of 2-D and 3-D crystals. However, the effect of wave amplitude on the deformed configuration can reveal valuable wave motion properties.
Therefore, the amplitude-dependent wave transmission should be explored for geometrically nonlinear phononic crystals.
List of Papers with Reference to Respective Chapters

Following are list of journal articles with reference to corresponding chapters:

Chapter 2


Chapter 3


Chapter 5

BIBLIOGRAPHY


