Instantaneous Optimal Control for Linear, Nonlinear and Hysteretic Structures – Stable Controllers

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J.N. Yang and Z. Li

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PREFACE

The National Center for Earthquake Engineering Research (NCEER) is devoted to the expansion and dissemination of knowledge about earthquakes, the improvement of earthquake-resistant design, and the implementation of seismic hazard mitigation procedures to minimize loss of lives and property. The emphasis is on structures and lifelines that are found in zones of moderate to high seismicity throughout the United States.

NCEER’s research is being carried out in an integrated and coordinated manner following a structured program. The current research program comprises four main areas:

- Existing and New Structures
- Secondary and Protective Systems
- Lifeline Systems
- Disaster Research and Planning

This technical report pertains to Program 2, Secondary and Protective Systems, and more specifically, to protective systems. Protective Systems are devices or systems which, when incorporated into a structure, help to improve the structure’s ability to withstand seismic or other environmental loads. These systems can be passive, such as base isolators or viscoelastic dampers; or active, such as active tendons or active mass dampers; or combined passive-active systems.

In the area of active systems, research has progressed from the conceptual phase to the implementation phase with emphasis on experimental verification. As the accompanying figure shows, the experimental verification process began with a small single-degree-of-freedom structure model, moving to larger and more complex models, and finally, to full-scale models.
This report is the latest in a series of NCEER technical reports addressing the development and application of instantaneous control algorithms to structural control. Within the framework of instantaneous optimal control, an important consideration is the assignment of the weighting matrix $Q$ which should be chosen to guarantee the stability of the controlled structure. A systematic way of assigning this weighting matrix is considered in this report and several possible choices are presented based on the Lyapunov direct method.
ABSTRACT

Recently, the instantaneous optimal control algorithms have been proposed and developed for applications to control of seismic-excited linear, nonlinear and hysteretic structural systems. In particular, these control algorithms are suitable for aseismic hybrid control systems for which the linear quadratic optimal control theory is not applicable. Within the framework of instantaneous optimal control, the weighting matrix $Q$ should be assigned to guarantee the stability of the controlled structure. A systematic way of assigning the weighting matrix by use of the Lyapunov direct method is investigated. Based on the Lyapunov method, several possible choices for the weighting matrix are presented, and their control performances are examined and compared for active and hybrid control systems under seismic loads. For the particular structures considered, the simplest choice for the $Q$ matrix seems to result in a good performance.
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SECTION 1
INTRODUCTION

Much progress has been made in active control of seismic-excited civil engineering structures, both in laboratory demonstrations and full-scale implementations [e.g., 1, 6, 7, 10, 11, 13, 14, 18]. It has been demonstrated recently that a combined use of active and passive control systems, referred to as the hybrid control system, can be more effective, beneficial and practical, in some cases, for seismic-excited buildings [e.g., 5, 9, 15, 19-26]. The application of aseismic hybrid control system involves active control of nonlinear or hysteretic structural systems, because most passive base isolation systems behave either nonlinearly or inelastically. The instantaneous optimal control algorithms proposed and developed by Yang et al [16-18] for applications to control of seismic-excited linear structures have been extended to control of nonlinear and hysteretic structures [19-26]. These control algorithms are particularly suited to aseismic hybrid control systems [e.g., 21-26, 3-4]. The performance index, J(t), for instantaneous optimal control algorithms is a time dependent quadratic function of the response and the control force [e.g., 16-26]. Optimal control for time dependent performance index is difficult to obtain, but it has several advantages as follows: (i) unlike the classical linear quadratic optimal control theory, the class of control forces is quite general for the time dependent performance index, which can include discontinuous or impulsive control forces, and (ii) it allows for approximations using the numerical solution formulation and, hence, it is applicable to control of nonlinear or hysteretic structural systems [e.g., 19, 21-25].

In the previous analyses [e.g., 16-25], the differential equations of motion were approximated by the state transition equations, which expressed the response of the structure at the time t in terms of the response at the previous time t-Δt. Such an approximation does not guarantee the stability of the controlled structure. The approximation is better as Δt become smaller; however, the control force, that is a linear function of Δt, should be finite [e.g., 16-25]. As a result, in addition to being positive semidefinite, the weighting matrix Q should be assigned to guarantee the stability of the controlled structure [26].

Frequently, one can assign the positive semidefinite weighting matrix Q and then check the stability condition by solving the eigenvalues of the controlled structure. This will involve
trial and error procedures. When the number of controllers is not small, this approach is feasible. For instance, if every floor of the building is installed with controllers, then a diagonal $Q$ matrix will be suitable. However, if the number of controllers is small compared to the number of degrees of freedom of the building, systematic methods to assign the $Q$ matrix in order to guarantee the stability of the controlled structure are highly desirable.

The purpose of this report is to present methods for determining the weighting matrix $Q$ to guarantee the structural stability using the Lyapunov direct method [e.g., 2, 8, 12]. With the application of the Lyapunov method, different possible choices for the weighting matrix $Q$ are investigated and their performances are examined and compared. Numerical examples are worked out to demonstrate the Lyapunov approach for applications to aseismic active and hybrid control systems that involve control of nonlinear or hysteretic structures. Particular choices for the weighting matrix are recommended, including the case which utilizes the velocity and acceleration feedbacks.
SECTION 2
FORMULATION FOR LINEAR STRUCTURES

2.1 Instantaneous Optimal Control With $Z(t)$ Feedback

For simplicity, consider a one-dimensional linear building structure equipped with an active control system as shown in Fig. 2-1 [11]. The structure is idealized by a lumped-mass n-degree-of-freedom system and subjected to a one-dimensional earthquake ground acceleration $\ddot{X}_0(t)$. The matrix equation of motion can be written as

$$
M \ddot{Y} + C \dot{Y} + K Y = H U + E \ddot{X}_0(t) \tag{2.1}
$$

in which $Y = [y_1, y_2, ..., y_n]' = \text{an n-vector with } y_j(t) \text{ being the relative displacement of the}$

$j\text{th floor with respect to the ground } X_0(t); M \text{ is an (nxn) diagonal mass matrix with the } j\text{th}$

diagonal element $m_j \text{ being the mass of the } j\text{th floor}; K \text{ and } C \text{ are (nxn) stiffness and damping}$

matrices, respectively. In Eq. (2.1), $U(t) = \text{a r-dimensional vector consisting of r control forces,}$

$H \text{ is an (nxr) location matrix and } E = -[m_1, m_2, ..., m_n]' \text{. In the notation above, an underbar}$

denotes either a vector or a matrix and a prime indicates the transpose of either a matrix or a vector.

The second order matrix equation, Eq. (2.1), can be converted into a first order equation by introducing a 2n state vector $Z(t)$

$$
Z(t) = \begin{bmatrix}
Y \\
\dot{Y}
\end{bmatrix}; \tag{2.2}
$$

with the result

$$
\dot{Z}(t) = A Z(t) + B U(t) + W_1 \ddot{X}_0(t) \tag{2.3}
$$

in which $A$ is a (2nx2n) system matrix, and $B$ and $W_1$ are (2nxr) matrix and 2n vector, respectively,

$$
A = \begin{bmatrix}
0 & I \\
-\frac{1}{M} K & -\frac{1}{M} C
\end{bmatrix} \tag{2.4}
$$

$$
B = \begin{bmatrix}
0 \\
M^{-1} H
\end{bmatrix}; \quad W_1 = \begin{bmatrix}
0 \\
M^{-1} E
\end{bmatrix} \tag{2.5}
$$
Fig. 2-1 : A Six-Story Building Equipped with Active Bracing System
Based on instantaneous optimal control, the time dependent objective function is given by [16, 17]

\[ J(t) = Z'(t)QZ(t) + U'(t)RU(t) \]  

(2.6)

in which \( Q \) is a \((2nx2n)\) positive semidefinite (symmetric) weighting matrix and \( R \) is a \((rxr)\) positive definite (symmetric) weighting matrix. When the weighting matrix \( Q \) is large, the response will be small and the required active control force \( U(t) \) will be large.

The optimal control vector can be expressed as [17]

\[ U(t) = -\phi R^{-1}B'QZ \]  

(2.7)

in which \( \phi = \Delta t/2 \) is a small positive constant. An alternative derivation for the optimal control vector \( U(t) \) for linear structures is presented in the Appendix I. In addition to being a positive semidefinite matrix, \( Q \) should also guarantee the stability of the controlled structure [26]. One possible way of choosing the \( Q \) matrix using the Lyapunov direct method is presented in the following [e.g., 2, 8, 12].

2.2 Lyapunov Direct Method

2.2.1 \( P_1 \) And \( P \) Matrices

Based on the Lyapunov direct method, a system defined by

\[ \dot{Z}(t) = \Delta Z(t) \]  

(2.8)

is stable if a Lyapunov function (scalar) \( V(Z) \rightarrow 0 \) for \( \dot{Z} \neq 0 \), \( V(Z) = 0 \) for \( \dot{Z} = 0 \), and \( V(Z) \rightarrow \infty \) as \( \dot{|Z|} \rightarrow \infty \) exists, such that its time derivative is negative semidefinite for all \( Z \), i.e., \( \dot{V}(Z) \leq 0 \). Consider a positive semidefinite matrix \( P_1 \), such that

\[ V(Z) = Z'P_1Z \geq 0 \]  

(2.9)

Taking the derivative of Eq. (2.9) and using Eq. (2.8), one obtains
\[ \dot{V} = Z' \mathbf{P}_1 Z + Z' \mathbf{P}_1 \dot{Z} = Z' (\mathbf{A}' \mathbf{P}_1 + \mathbf{P}_1 \mathbf{A}) Z \] (2.10)

For \( \dot{V} \) to be negative semidefinite, the matrix \( \mathbf{A}' \mathbf{P}_1 + \mathbf{P}_1 \mathbf{A} \) must be negative semidefinite, i.e.,

\[ \mathbf{A}' \mathbf{P}_1 + \mathbf{P}_1 \mathbf{A} = -\mathbf{I}_0 \] (2.11)

in which \( \mathbf{I}_0 \) is any (symmetric) positive semidefinite matrix. The positive semidefinite matrix \( \mathbf{P}_1 \) can easily be solved from Eq. (2.11), since it is a linear matrix equation [e.g., 2, 8, 12]. Furthermore, the stationary matrix Riccati equation is given by [e.g., 2, 8, 12]

\[ \mathbf{A}' \mathbf{P} + \mathbf{P} \mathbf{A} - \mathbf{P} \mathbf{R} \mathbf{R}^{-1} \mathbf{B}' \mathbf{P} = -\mathbf{I}_0 \] (2.12)

in which \( \mathbf{P} \) is the \((2n \times 2n)\) stationary Riccati matrix that is positive semidefinite.

Consider the structural system defined by Eq. (2.3) with the control vector \( \mathbf{U}(t) \) given by Eq. (2.7). Then, the equation of motion is given by

\[ \dot{\mathbf{Z}} = (\mathbf{A} - \phi \mathbf{R} \mathbf{R}^{-1} \mathbf{B}' \mathbf{Q}) \mathbf{Z} \] (2.13)

in which the excitation \( \mathbf{W}_1 \ddot{X}_0(t) \) is dropped because it is not relevant to the stability of the structure. Likewise, the structure without control is stable, i.e., the real parts of all eigenvalues of the matrix \( \mathbf{A} \) are negative.

A possible Lyapunov function is given as follows

\[ V(Z) = Z' \mathbf{Q} Z \geq 0 \] (2.14)

from which

\[ \dot{V} = Z' \mathbf{Q} Z + Z' \mathbf{Q} \dot{Z} = Z' (\mathbf{A}' \mathbf{Q} + \mathbf{Q} \mathbf{A} - 2 \phi \mathbf{Q} \mathbf{R} \mathbf{R}^{-1} \mathbf{B}' \mathbf{Q}) Z \] (2.15)

Since \( \mathbf{R} \) is positive definite, \( \mathbf{R}^{-1} \) is positive definite and \( \mathbf{BR}^{-1} \mathbf{B}' \) is positive semidefinite. Hence, the term \(-2\phi \mathbf{Q} \mathbf{R} \mathbf{R}^{-1} \mathbf{B}' \mathbf{Q}\) is negative semidefinite. Thus, it follows from Eq. (2.15) that the control system is stable if \( \mathbf{A}' \mathbf{Q} + \mathbf{Q} \mathbf{A} \) is negative semidefinite, i.e.,
\[ A'Q + QA = -I_0 \] (2.16)

A comparison of Eqs. (2.11) and (2.16) indicates that the matrix \( P_1 \) is a possible choice for the \( Q \) matrix. In a similar manner, it can be shown that \( \phi_1 P_1 \) is also a possible choice of the \( Q \) matrix where \( \phi_1 > 0 \).

Usually, the \( Q \) matrix can be expressed for convenience as

\[ Q = \phi_2 Q_1 \] (2.17)

in which \( \phi_2 \) is a positive number. Substitution of Eq. (2.17) into Eq. (2.15) yields

\[ \dot{V} = \phi_2 Z' \left[ A'Q_1 + Q_1 A - 2\phi \phi_2 Q_1 B R^{-1} B'Q_1 \right] Z \] (2.18)

For \( \dot{V} \leq 0 \), one obtains from Eq. (2.18)

\[ A'Q_1 + Q_1 A - 2\phi \phi_2 Q_1 B R^{-1} B'Q_1 = -I_0 \] (2.19)

Equation (2.19) is a Riccati-type matrix equation from which \( Q_1 \) can be solved. Furthermore, if \( \phi_2 \) is chosen to be 0.5/\( \phi \) such that \( 2\phi \phi_2 = 1 \) (or \( \phi \phi_2 = 0.5 \)), then Eq. (2.19) is exactly the matrix Riccati equation given by Eq. (2.12) and hence \( Q_1 = P \). Further, it can also be shown that \( \phi_3 P \) is also a possible choice for the \( Q_1 \) matrix for \( \phi_3 > 1 \). In addition to \( 2\phi \phi_2 = 1 \), if \( \phi_3 \) is chosen to be 2.0, i.e., \( \phi_3 = 2 \), then the control vector, Eq. (2.7), becomes \( U(t) = -R^{-1}B'PZ(t) \).

This is exactly the result of linear quadratic optimal control. Consequently, linear quadratic optimal control is a special case of the approximate solution for instantaneous optimal control.

The conclusions derived above indicate that \( \phi_1 P_1 \), \( \phi_3 P \) and \( \phi_2 Q_1 \) are possible choices for the \( Q \) matrix where \( \phi_1, \phi_2 \) and \( \phi_3 \) are positive numbers [26].

2.2.2 Energy Consideration

The Lyapunov approach is based on the generalization of the energy concept. In order
to provide guidances for the selection or assignment of the \( Q \) matrix, the energy concept will be considered. Unlike the Riccati matrix \( P \) that involves the solution of \((2n \times 2n)\) nonlinear equations, the \( P_1 \) matrix involves the solution of linear algebraic equations, Eq. (2.11). Hence, Eq. (2.16) will be examined further.

The \((2n \times 2n)\) positive semidefinite matrices \( Q \) and \( I_0 \) are partitioned into four \((n \times n)\) submatrices

\[
Q = \begin{bmatrix} Q_{11} & Q_{12} \\ - & - \\ Q'_{12} & Q_{22} \end{bmatrix} ; \quad I_0 = \begin{bmatrix} I_1 & 0 \\ - & - \\ 0 & I_2 \end{bmatrix}
\tag{2.20}
\]

in which \( Q_{11} \), \( Q_{12} \), \( Q_{22} \), \( I_1 \) and \( I_2 \) are \((n \times n)\) symmetric matrices.

The Lyapunov function \( V(Z) = Z'QZ \), Eq. (2.14), can be chosen as the total energy of the structure, i.e., the sum of the potential and kinetic energies. In this case, the \( Q \) matrix given by Eq. (2.20) becomes

\[
Q_{11} = \phi_1 K \quad ; \quad Q_{22} = \phi_1 M \quad ; \quad Q_{12} = 0
\tag{2.21}
\]

in which \( \phi_1 > 0 \). Substituting Eq. (2.21) into the \( Q \) matrix given by Eq. (2.20) and then into the left hand side of Eq. (2.16), one can easily show that \( A'QQ'A \) is a negative semidefinite matrix. The resulting negative semidefinite matrix is given by \(-I_0\) in the form of Eq. (2.20) in which \( I_1 = 0 \) and \( I_2 = 2C\phi_1 \).

Thus, the derivative of the Lyapunov function is given by \( \dot{V} = Z'(A'QQ'A)Z = -2\phi_1 \dot{X}'C\dot{X} \) which is the energy dissipation of the structure. Consequently, Eq. (2.21) is another choice for the \( Q \) matrix in which the Lyapunov function is the total energy of the structure.
The choice of the $Q$ matrix, based on the energy concept given by Eq. (2.21), is quite simple. Further, since $Q_{12} = 0$ and $Q_{11}$ does not have any contribution to the control vector $U(t)$, no displacement sensor (or measurement) is needed. In civil engineering applications, it is not feasible to install many controllers. If $m$ controllers are used, then since $Q_{22} = \phi_1 M$ is a diagonal matrix, only the information from $m$ velocity sensors that are co-located with the controllers are relevant to the control vector $U(t)$. Consequently, the performance of the control system may or may not be satisfactory if the number of controllers is small. However, if many controllers are used, the total energy, Eq. (2.21), is a satisfactory choice. For some types of aseismic hybrid control systems, the simple choice for the $Q$ matrix based on the energy concept, Eq. (2.21), is very satisfactory. This will be demonstrated later in a numerical example.

Based on the total energy consideration, Eq. (2.21), the matrix $A'Q + QA$ is shown to be a negative semidefinite matrix $-I_0$ given by Eq. (2.20) where $I_1 = 0$ and $I_2 = 2\phi_1 Q$. A generalization of the energy concept is to choose $Q_{12} \neq 0$, but still let $I_1 = 0$. Let $P_v$ be a choice of $Q$ such that

$$A'P_v + P_v A = -I_0$$  \hspace{1cm} (2.22)

in which $I_0$ is given by Eq. (2.20) with $I_1 = 0$ and $I_2$ being a diagonal matrix. It will be shown in the numerical examples that such a choice is quite reasonable.

### 2.3 Instantaneous Optimal Control With $\ddot{Z}(t)$ Feedback

For instantaneous optimal control with velocity and acceleration feedbacks, the time dependent performance index is given by [25]
\[ J^*(t) = \dot{Z}(t) Q^* \dot{Z}(t) + U'(t)R U(t) \]  
(2.23)

and the optimal control vector is given by

\[ U(t) = - R^{-1} B'Q^* \dot{Z}(t) \]  
(2.24)

To determine the \( Q^* \) matrix for guaranteeing the structural stability using the Lyapunov approach, the following transformation is made,

\[ Q^* = -(A^{-1})'P^* = - \tilde{A}'P^* ; \quad \tilde{A} = A^{-1} \]  
(2.25)

in which \( P^* \) is a \((2nx2n)\) positive semidefinite (symmetric) matrix to be determined. Substituting Eqs. (2.24) and (2.25) into Eq. (2.3), the equations of motion without the excitation \( \ddot{X}_0(t) \) becomes

\[ Z(t) = \tilde{A} [I_4 - B R^{-1}B' \tilde{A}'P^*] \dot{Z}(t) \]  
(2.26)

in which \( I_4 \) is a \((2nx2n)\) identity matrix.

The Lyapunov function is defined by \( V(Z) = Z'P^*Z \geq 0 \) and hence

\[ V = Z'P^*Z + Z'P^*Z = Z'[\tilde{A}'P^* + P^*\tilde{A} - 2P^*(\tilde{A}B)R^{-1}(\tilde{A}B)'P^*]Z(t) \]  
(2.27)

in which Eq. (2.26) has been used. For \( \dot{V} \leq 0 \), \( P^* \) must satisfy the following matrix equation

\[ \tilde{A}'P^* + P^*\tilde{A} - 2P^*(\tilde{A}B)R^{-1}(\tilde{A}B)'P^* = -I_0 \]  
(2.28)

Equation (2.28) is precisely the Riccati-type matrix equation, Eq. (2.19), with system matrices \( A^* \) and \( B^* \) given by

\[ A^* = \tilde{A} = A^{-1} \text{ and } B^* = \tilde{A}B = A^{-1}B \]  
(2.29)

Since the uncontrolled structure is stable, the real parts of the eigenvalues of \( A \) are negative and hence the real parts of the eigenvalues of \( A^{-1} \) are also negative. Thus, Eq. (2.28) guarantees a positive definite matrix \( P^* \). The numerical techniques for solving the Riccati-type matrix equation are given, e.g., in Ref. 2.
In a similar manner, since $-P^*(\tilde{A}B)R^{-1}(\tilde{A}B)'P^*$ is negative semidefinite, the $P^*$ matrix can also be obtained either from the Riccati matrix equation

$$\tilde{A}'P^* + P^*\tilde{A} - P^*(\tilde{A}B)R^{-1}(\tilde{A}B)'P^* = -I_0$$

(2.30)

or from the linear matrix equation

$$\tilde{A}'P^* + P^*\tilde{A} = -I_0$$

(2.31)

After the $P^*$ matrix is determined either from Eq. (2.28) or Eq. (2.30) or Eq. (2.31), the $Q^*$ matrix is obtained from Eq. (2.25). Note that $Q^*$ obtained from Eq. (2.25) is positive semidefinite but not symmetric.
SECTION 3

FORMULATION FOR NONLINEAR STRUCTURES: HYBRID CONTROL

3.1 Equation of Motion and Lyapunov Function

When the structure system is nonlinear or hysteretic in nature, such as the building equipped with a hybrid control system as shown in Fig. 3-1, the matrix equation of motion can be written as [21, 24]

\[ \ddot{Z}(t) = g[Z] + B \ U(t) + W_1 \ \ddot{X}_0(t) \]  \hspace{1cm} (3.1)

in which \( B \) and \( W_1 \) are given by Eq. (2.5), and \( Z(t)=[X', \dot{X}']' \) is a 2n state vector with \( X=[x_1, x_2, \ldots, x_n]' \) where \( x_i \) is the interstory deformation of the \( i \)th story unit. Although \( B \) and \( W_1 \) are given by Eq. (2.5), they are different from those for the linear structures, because the coordinate has been changed from \( Y \) to \( X \) [see 21, 24]. In Eq. (3.1), \( g[Z(t)] \) is a 2n vector that is a nonlinear function of the damping vector, \( F_D[\dot{X}] \), and stiffness vector \( F_S[X] \) given by [21, 24]

\[ g[Z] = \begin{bmatrix} \dot{X} \\ -M^{-1}_1 [F_D(\dot{X}) + F_S(X)] \end{bmatrix} \]  \hspace{1cm} (3.2)

The instantaneous optimal control vector \( U(t) \) is given by Eq. (2.7), i.e., \( U(t) = -\phi R^{-1} B' Q Z \), where \( \phi = \Delta t/3 \). Further, Eq. (2.17) will be used in the following, i.e., \( Q = \phi_2 Q_1 \). Substituting Eq. (2.7) into Eq. (2.17) and then into Eq. (3.1), one obtains the matrix equation of motion for the controlled structure

\[ \ddot{Z}(t) = g(Z) - \phi_2 BR^{-1} B' Q_1 Z(t) \]  \hspace{1cm} (3.3)

in which the excitation \( \ddot{X}_0(t) \) has been dropped.

The following Lyapunov function is considered
Fig. 3-1: Structural Model of a Multi-Story Building Equipped with Aseismic Hybrid Control Systems; (a) Rubber Bearing Isolation System and Actuator; (b) Rubber Bearing Isolation System and Active Mass Damper
\[ V(Z) = \dot{Z}(t)Q_1 \ddot{Z}(t) \]  

(3.4)

Hence

\[ \dot{V} = \dot{Z}(t)Q_1 \ddot{Z}(t) + \ddot{Z}(t)Q_1 \dot{Z}(t) \]  

(3.5)

Taking the derivative of Eq. (3.3), one obtains the matrix equation of motion as

\[ \ddot{Z}(t) = [\Delta(Z) - \phi \phi_2 B R^{-1} B' Q_1] \dot{Z}(t) \]  

(3.6)

in which

\[ \Delta(Z) = \frac{\partial g(Z)}{\partial Z} \]  

(3.7)

is a (2nx2n) matrix and the relation \( \dot{g}(Z) = (\frac{\partial g(Z)}{\partial Z})(\frac{\partial Z}{\partial t}) = \Delta(Z) \dot{Z} \) has been used. Substitution of Eq. (3.6) into Eq. (3.5) leads to the expression

\[ \dot{V} = \dot{Z}(t)[\Delta'(Z)Q_1 + Q_1 \Delta(Z) - 2 \phi \phi_2 Q_1 B R^{-1} B' Q_1] \dot{Z}(t) \]  

(3.8)

For \( \dot{V} \leq 0 \), one obtains

\[ \Delta'(Z)Q_1 + Q_1 \Delta(Z) - 2 \phi \phi_2 Q_1 B R^{-1} B' Q_1 = -I_0 \]  

(3.9)

This is precisely the Riccati-type matrix equation for the determination of \( Q_1 \). Since the system without control is stable, the real parts of the eigenvalues of \( \Delta(Z) \) are negative, and hence \( Q_1 \) obtained from Eq. (3.9) is positive semidefinite. Furthermore, since \( Q_1 B R^{-1} B' Q_1 \) is positive semidefinite, \( Q_1 \) can also be obtained from the following linear algebraic equation

\[ \Delta'(Z)Q_1 + Q_1 \Delta(Z) = -I_0 \]  

(3.10)

3.2 Equivalent Linearization for Determination of Constant Q Matrix

It is observed from Eqs. (3.7) and (3.9)-(3.10) that the nonlinear system matrix \( \Delta(Z) = \frac{\partial g(Z)}{\partial Z} \) is a function of the response vector \( Z(t) \) that in turn is a function of time \( t \). As a result, the weighting matrix \( Q \) is a function of time \( t \). In other words, the weighting matrix \( Q \) should be solved at every time instant \( t \) when the state vector \( Z(t) \) is measured. This may
present a problem for practical implementations of the control system, since $Q$ has to be computed on-line at every time instant $t$. Consequently, two equivalent linearization techniques are presented in the following so that the system matrix $\Delta(Z)$ can be approximated reasonably by a constant matrix, resulting in a constant $Q_1$ matrix.

The first approximation is to linearize the structural properties at either the equilibrium position $Z=0$ or other suitable point $Z=Z_0$ depending on the characteristics of the nonlinear or hysteretic structure. Such an approximation may or may not be reasonable; however, the final result should be verified by simulation. For aseismic hybrid control systems using a combination of a rubber bearing isolation system and active control devices, the linearization at the equilibrium position seems to be reasonable, i.e.,

$$\Delta = \Delta(Z) \big|_{Z=0} \quad (3.11)$$

Another approach is to linearize the structural properties based on the physical consideration. For hysteretic structures, the equivalent linear stiffness and damping can be obtained from the integration of the hysteresis loop. This approach will be presented elsewhere due to space limitation.

For the hysteretic structure equipped with a passive base isolation system, the stiffness restoring force, $F_{si}(t)$, of the $i$th story, including the base isolation system, can be expressed [21-24]

$$F_{si}(t) = \alpha_i k_i x_i + (1 - \alpha_i) k_i D_{yi} \gamma_i \quad (3.12)$$

in which $k_i=$elastic stiffness of the $i$th story unit, $\alpha_i=$ratio of post-yielding to pre-yielding stiffness, $D_{yi}=$yield deformation=constant, and $\gamma_i$ is a nondimensional variable introduced to describe the hysteretic component of the deformation, with $|\gamma_i| \leq 1$, where
\[ \ddot{\mathbf{v}}_i = D_{\mathbf{v}}^{-1} [A_i \dot{\mathbf{x}}_i - \beta_i |\dot{\mathbf{x}}_i| |\mathbf{v}_i|^{n_i-1} \mathbf{v}_i - \gamma_i \dot{\mathbf{x}}_i |\mathbf{v}_i|^{n_i}] = f_i(\dot{\mathbf{x}}_i, \mathbf{v}_i) \]  

(3.13)

In Eq. (3.13), parameters \( A_i, \beta_i \) and \( \gamma_i \) govern the scale and general shape of the hysteresis loop, whereas the smoothness of the force-deformation curve is determined by the parameter \( n_i \).

With such a hysteretic model for the stiffness restoring force and assuming a linear viscous damping, the system matrix \( \Delta \) linearized at \( \mathbf{Z} = \mathbf{0} \) and \( \mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]' = \mathbf{0} \) can be obtained as follows [see Appendix II],

\[
\Delta = \begin{bmatrix}
\mathbf{Q} & I \\
-\mathbf{M}_1^{-1} \mathbf{C}_1 & -\mathbf{M}_1^{-1} \mathbf{K}_1
\end{bmatrix}
\]  

(3.14)

in which \( \mathbf{I} \) is an (nxn) identity matrix; \( \mathbf{M}_1 \) is an (nxn) mass matrix with the i-jth element \( \mathbf{M}_1(i,j) = m_i \) for \( i = 1, 2, \ldots, n \) and \( j = 1, 2, \ldots, i \), and \( \mathbf{M}_1(i,j) = 0 \) for \( j > i \), where \( m_i \) is the mass of the ith floor; \( \mathbf{C}_1 \) and \( \mathbf{K}_1 \) are (nxn) band-limited damping and elastic stiffness matrices with all elements equal to zero except \( \mathbf{C}_1(i,i) = c_i \), \( \mathbf{K}_1(i,i) = k_i \) for \( i = 1, 2, \ldots, n \) and \( \mathbf{C}_1(i,i+1) = -c_{i+1} \), \( \mathbf{K}_1(i,i+1) = -k_{i+1} \) for \( i = 1, 2, \ldots, n-1 \), where \( c_i \) and \( k_i \) are the damping coefficient and the elastic stiffness, respectively, of the ith story unit. Matrices \( \mathbf{M}_1, \mathbf{C}_1 \), and \( \mathbf{K}_1 \) are referred to the coordinate system \( \mathbf{X}(t) \). Elements of \( \mathbf{M}_1, \mathbf{C}_1 \) and \( \mathbf{K}_1 \) matrices described above hold for the hybrid control system shown in Fig. 3-1(a). For the hybrid control system shown in Fig. 3-1(b), these elements should be modified appropriately.

Thus, the determination of the \( \mathbf{Q} \) (or \( \mathbf{Q}_i \)) matrix for nonlinear or hysteretic structural systems can be made in the same way as that for the linear structure described previously, except that the system matrix \( \Delta \) given by Eq. (3.14) should be used instead of \( \Delta \) given by Eq. (2.4).
SECTION 4
DEMONSTRATIVE EXAMPLES

4.1 Linear Structure

To demonstrate the performance of various choices for the \( Q \) matrix and to compare the performance with each other, a six-story full-scale linear test building, Fig. 2-1, recently constructed in Tokyo, Japan by Takenaka Company is considered first [1, 11]. The building is equipped with an active bracing system on the first floor and an active mass damper on the top floor. The mass of each floor is identical and is equal to 100 metric tons. The natural frequencies of the building are computed as 0.943, 2.765, 4.876, 7.279, 10.114 and 14.423 Hz. The damping ratio for each vibrational mode is 1%. The El Centro earthquake ground acceleration scaled by a factor of 32\% as shown in Fig. 4-1(a) is used as the input excitation [11]. Without any control system, the maximum interstory deformation \( x_i (i=1,2,\ldots,6) \) of each story unit, the maximum total acceleration \( a_i \) of each floor and the maximum relative displacement \( y_6 \) of the top floor with respect to the ground are shown in the columns, designated as "No Control", of Table 4-1.

With the active bracing system (ABS) in which the angle of inclination \( \theta \) of the active bracing is 51.5\°, the building response and the active control force depend on the particular choice of the \( Q \) matrix. For the present example with only one controller, the \( R \) matrix consists of only one element, denoted by \( R_0 \). For demonstrative purpose, \( R_0 = 0.01 \) and \( \phi = \Delta t/2 = 0.01 \) are used. For simplicity, the \( I_1 \) and \( I_2 \) matrices, Eq. (2.20), are considered to be diagonal matrices with diagonal elements \( I_{1ii} \) and \( I_{2ii} \) (\( i=1,2,\ldots,n \)), respectively. The diagonal elements in each of the \( I_1 \) and \( I_2 \) matrices are identical, i.e., \( I_{1ii} = I_{1kk} \) and \( I_{2ii} = I_{2kk} \) for \( i,k = 1,2,\ldots,n \).
Fig. 4-1: Earthquake Ground Acceleration; (a) 32% El Centro Earthquake; (b) A Simulated Earthquake
Table 4-1: Maximum Response Quantities of A Six-Story Building for Different Choices of the Weighting Matrix

| STORY NO | x<sub>i</sub> cm | a<sub>i</sub> cm/s<sup>2</sup> | x<sub>i</sub> cm | a<sub>i</sub> cm/s<sup>2</sup> | x<sub>i</sub> cm | a<sub>i</sub> cm/s<sup>2</sup> | x<sub>i</sub> cm | a<sub>i</sub> cm/s<sup>2</sup> | x<sub>i</sub> cm | a<sub>i</sub> cm/s<sup>2</sup> | x<sub>i</sub> cm | a<sub>i</sub> cm/s<sup>2</sup> |
|----------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 1        | 0.733 cm        | 120.21 cm/s<sup>2</sup> | 0.533 cm        | 192.92 cm/s<sup>2</sup> | 0.477 cm        | 182.09 cm/s<sup>2</sup> | 0.303 cm        | 566.14 cm/s<sup>2</sup> | 0.417 cm        | 173.13 cm/s<sup>2</sup> | 0.416 cm        | 172.82 cm/s<sup>2</sup> |
| 2        | 1.453 cm        | 234.26 cm/s<sup>2</sup> | 0.558 cm        | 125.97 cm/s<sup>2</sup> | 0.556 cm        | 136.06 cm/s<sup>2</sup> | 0.992 cm        | 130.28 cm/s<sup>2</sup> | 0.573 cm        | 162.68 cm/s<sup>2</sup> | 0.575 cm        | 159.36 cm/s<sup>2</sup> |
| 3        | 1.613 cm        | 283.28 cm/s<sup>2</sup> | 0.527 cm        | 148.15 cm/s<sup>2</sup> | 0.577 cm        | 156.82 cm/s<sup>2</sup> | 1.240 cm        | 177.67 cm/s<sup>2</sup> | 0.675 cm        | 168.58 cm/s<sup>2</sup> | 0.678 cm        | 168.45 cm/s<sup>2</sup> |
| 4        | 1.751 cm        | 284.15 cm/s<sup>2</sup> | 0.567 cm        | 155.3 cm        | 0.628 cm        | 172.44 cm/s<sup>2</sup> | 1.095 cm        | 203.64 cm/s<sup>2</sup> | 0.742 cm        | 199.80 cm/s<sup>2</sup> | 0.742 cm        | 199.46 cm/s<sup>2</sup> |
| 5        | 1.648 cm        | 331.48 cm/s<sup>2</sup> | 0.638 cm        | 136.89 cm        | 0.722 cm        | 154.70 cm/s<sup>2</sup> | 1.029 cm        | 216.55 cm/s<sup>2</sup> | 0.780 cm        | 171.03 cm/s<sup>2</sup> | 0.780 cm        | 170.09 cm/s<sup>2</sup> |
| 6        | 1.133 cm        | 360.77 cm/s<sup>2</sup> | 0.561 cm        | 202.53 cm        | 0.628 cm        | 226.15 cm/s<sup>2</sup> | 0.778 cm        | 266.81 cm/s<sup>2</sup> | 0.689 cm        | 250.59 cm/s<sup>2</sup> | 0.689 cm        | 250.73 cm/s<sup>2</sup> |
| \( y_6 \) | 7.98 cm        | 2.47 cm        | 2.68 cm        | 4.79 cm        | 2.97 cm        | 2.98 cm        | 2.69 cm        |
The first case for the $Q$ matrix is $Q=\phi_2Q_1$ presented in Eq. (2.17), where $Q_1$ is computed from the Riccati-type matrix equation, Eq. (2.19). $\phi_2$ is chosen to be 100 such that $\phi\phi_2=1$, whereas $I_{1ii}=255\times10^5$ and $I_{2ii}=255\times10^5$. During the entire earthquake episode, the maximum building response quantities are shown in the columns, designated as "$\phi_2Q_1$", of Table 4-1. Also shown in Table 4-1 are the required maximum control force $U_{max}$. The second case considered for the $Q$ matrix is exactly the Riccati matrix. In this case, $\phi\phi_2=0.5$ and $\phi_2=2$ so that the control force, Eq. (2.7), is $U(t)=-R^{-1}P'PZ$ where $P$ is computed from Eq. (2.12). $I_{1ii}$ and $I_{2ii}$ are identical to the previous case. The maximum response quantities are presented in the columns, designated as "$P$", of Table 4-1.

The next choice for $Q$ is the energy consideration given by Eqs. (2.20) and (2.21) in which $\phi_1=1.5\times10^5$. The corresponding results are presented in the columns, designated as "$P_e$", of Table 4-1. We next consider the choice of $P_1$ matrix in which $\phi_1=1.0$, $I_{1ij}=1.63\times10^4$ and $I_{2ij}=1.63\times10^4$. The $Q$ matrix is computed from Eq. (2.16) with $I_0$ given by Eq. (2.20). The maximum building response quantities are shown in the columns, designated as "$P_1$", of Table 4-1.

Another choice of $Q$ is the $P_V$ matrix computed from Eqs. (2.22) and (2.20) with $I_1=0$ and $I_{2ij}=520\times10^5$. The maximum response quantities are shown in the columns, designated as "$P_V$", of Table 4-1.

Finally we consider the case for $\dot{Z}(t)$ feedback, Eqs. (2.23) and (2.24). The $Q^*$ matrix is obtained from Eq. (2.25), whereas the $P^*$ matrix is computed from Eq. (2.30). In the present case $R_0=10^{-5}$, $I_{1ii}=5\times10^3$ and $I_{2ii}=0$, see Eq. (2.20). The maximum response quantities are
presented in the columns, designated as "P*", of Table 4-I.

Different choices of the Q matrix presented in Table 4-I were made in such a way that the required maximum control force is almost the same, i.e., about 1880 kN. Since the maximum interstory deformation of each story unit, x_j, and the maximum acceleration of each floor, a_i, are important safety measures, they are presented in Table 4-I. It is observed from Table 4-I that the choice of the Q matrix using the Riccati-type equation, denoted by φ_2Q_1, and the exact Riccati equation, denoted by P, are quite satisfactory. For this particular example, the straight-forward application of the total energy concept, denoted by P_e, is less satisfactory. However, the choice of the P_1 matrix, denoted by P_1, and the choice of the P_Y matrix, denoted by P_Y, are very satisfactory comparing with the choices of φ_2Q_1 and P. In these cases the computation involves only the numerical solution of linear algebraic equations. Furthermore, the application of the Z feedback using the Riccati-type solution for the Q* matrix is also very satisfactory as shown in the last two columns of Table 4-I.

The observations made above are based on the results presented in Table 4-I. Strictly speaking, the comparison of the performance of various choices for the Q matrix should be based on the time dependent performance index J(t) given by Eq. (2.6). In this connection, it is mentioned that the selection of the Q matrix using the Riccati matrix seems to result in a smaller performance index J(t) over a significant range of the time domain t.

4.2 Hybrid Control

An eight-story building that exhibits bilinear elasto-plastic behavior is considered [e.g., 21-24]. The stiffness of each story unit is designed such that yielding occurs simultaneously for
each story unit. The properties of the building are as follows: (i) the mass of each floor is identical with \( m_i = m = 345.6 \text{ metric tons} \); (ii) the preyielding stiffnesses of the eight-story units are \( k_{i1} \) (i=1,2,...,8) = 3.4x10^5, 3.26x10^5, 2.85x10^5, 2.69x10^5, 2.43x10^5, 2.07x10^5, 1.69x10^5 and 1.37x10^5 kN/m, respectively, and the postyielding stiffnesses are \( k_{i2} = 0.1 \) \( k_{i1} \) for i=1,2,...,8, i.e., \( \alpha_i = 0.1 \) and \( k_i = k_{i1} \); and (iii) the viscous damping coefficients for each story unit are \( c_i = 490, 467, 410, 386, 348, 298, 243 \) and 196 kN.sec/m, respectively. The damping coefficients given above result in a classically damped building with a damping ratio of 0.38\% for the first vibrational mode. The natural frequencies of the unyielded building are 5.24, 14.0, 22.55, 30.22, 36.89, 43.06, 49.54 and 55.96 rad./sec. The yielding level for each story unit varies with respect to the stiffness; with the results, \( D_{yi} = 2.4, 2.3, 2.2, 2.1, 2.0, 1.9, 1.7, \) and 1.5 cm. The bilinear elasto-plastic behavior can be described by the hysteretic model, Eqs. (3.12) and (3.13), with \( A_i = 1.0, \beta_i = 0.5, n_i = 95 \) and \( \gamma_i = 0.5 \) for i=1,2,...,8 [24]. A simulated earthquake with a maximum ground acceleration of 0.3g as shown in Fig. 4-1(b) is used as the input excitation [24].

Within 30 seconds of the earthquake episode, the maximum interstory deformation, \( x_i \), and the maximum absolute acceleration of each floor, \( a_i \), are shown in Table 4-II, designated as "No Control". As observed from Table 4-II, the deformation of the unprotected building is excessive and that yielding takes place in each story unit, i.e., \( x_i > D_{yi} \) [24].

To reduce the structural response, a lead-core rubber-bearing isolation system is implemented as shown in Fig. 3-1(a). The restoring force of the lead-core rubber-bearing system is modeled by Eq. (3.12) with \( F_{sb} = \alpha_b k_b x_b + (1 - \alpha_b) k_b D_{yb} v_b \) in which \( v_b \) is given by Eq. (3.13) with i=b. The mass of the base isolation system is \( m_b = 450 \text{ metric tons} \) and the viscous
Table 4-II: Maximum Response Quantities of A Base-Isolated Building

<table>
<thead>
<tr>
<th>STORY NO</th>
<th>D_y (cm)</th>
<th>( \alpha )</th>
<th>( x_i ) cm</th>
<th>( a_i ) cm/s²</th>
<th>( x_i ) cm</th>
<th>( a_i ) cm/s²</th>
<th>( x_i ) cm</th>
<th>( a_i ) cm/s²</th>
<th>( x_i ) cm</th>
<th>( a_i ) cm/s²</th>
<th>( x_i ) cm</th>
<th>( a_i ) cm/s²</th>
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<td>42.7</td>
<td>158</td>
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<td>137.5</td>
<td>24.9</td>
<td>113.6</td>
<td>24.9</td>
<td>110.8</td>
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<td>113.6</td>
<td>24.9</td>
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<td>24.9</td>
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<td>24.9</td>
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<td>24.9</td>
<td>113.6</td>
<td>24.9</td>
<td>110.8</td>
<td>27.25</td>
</tr>
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</table>
Fig. 4-2: Hysteretic Characteristics of Base Isolation System
damping coefficient is assumed to be linear with \( c_b = 26.17 \text{ kN.sec/m} \). The restoring force of the base isolation system given above is not bilinear elasto-plastic and the parameter values are given as follows: \( k_b = 18050 \text{ kN/m} \), \( \alpha_b = 0.6 \), \( D_y = 4 \text{ cm} \), \( A_b = 1.0 \), \( \beta_b = 0.5 \), \( n_b = 3 \) and \( \gamma_b = 0.5 \), Eq. (3.13). The hysteresis loop of such a base isolation system, i.e., \( x_b \) versus \( v_b \), is shown in Fig. 4-2. For the building with the base isolation system alone, the first natural frequency of the preyielded structure is 2.21 rad/sec and the damping ratio for the first vibrational mode is 0.15\%. The response vector \( \mathbf{X}(t) \) is given by \( \mathbf{X} = [x_b, x_1, \ldots, x_8]' \).

The maximum response quantities of the building in 30 seconds of the earthquake episode are shown in the columns of Table 4-II designated as "With BIS". As observed from Table 4-II, the interstory deformation and the floor acceleration are drastically reduced. However, the deformation of the base isolation system shown in row B of Table 4-II may be excessive.

To protect the safety and integrity of the base isolation system, an actuator is connected to the base isolation system as shown in Fig. 3-1(a). With the actuator applying the active control force \( \mathbf{U}(t) \) to the base isolation system, the structural response depends on the weighting matrices \( \mathbf{R} \) and \( \mathbf{Q} \). For this example, the weighting matrix \( \mathbf{R} \) consists of only one element, denoted by \( R_0 \). For illustrative purpose, \( \phi = \Delta t/3 = 0.0067 \) and \( R_0 = 0.0067 \) are used.

First, we consider the choice of \( \mathbf{P}_V \) for the \( \mathbf{Q} \) matrix, Eq. (2.22), in which \( I_1 = \mathbf{Q} \) and the diagonal elements of the diagonal matrix \( I_2 \) are \( I_{2j} = 1.02 \times 10^5 (j = 1, 2, \ldots, 9) \). The maximum building response quantities and the maximum control force \( U_m \) are presented in the columns, designated by "\( \mathbf{P}_V \)", of Table 4-II. Another possible choice considered for the \( \mathbf{Q} \) matrix is \( \phi_2 Q_1 \) where \( Q_1 \) is computed from the Riccati-type equation, Eq. (2.19), and \( \phi_2 \) is chosen to be 150.
The $I_0$ matrix used has the form given by Eq. (2.20) with $I_1 = Q$ and all elements of $I_2$ are zero except $I_2(1,1) = 2.5 \times 10^5$. Within 30 seconds of the earthquake episode, the maximum response quantities are shown in the columns, designated as "$\phi_2 Q_1$", of Table 4-II.

Finally, the energy approach is considered in which the $Q$ matrix is given by Eqs. (2.20) and (2.21), i.e., $Q_{11} = \phi_1 K$, $Q_{22} = \phi_1 M$ and $Q_{12} = Q$. The total energy expression above is referred to the coordinate $\mathbf{Y}$. For nonlinear or hysteretic structures in which the coordinate $\mathbf{X}$ (interstory deformation) is used, a transformation $\mathbf{Y} = \mathbf{T} \mathbf{X}$ can be made where $\mathbf{T}$ is the transformation matrix. Through such a transformation, the $Q$ matrix in the $\mathbf{X}$ coordinate can still be expressed in the form of Eq. (2.20) with

$$Q_{11} = \phi_1 \mathbf{T} \mathbf{K}, \quad Q_{22} = \phi_1 \mathbf{T} \mathbf{M}, \quad Q_{12} = Q_{21} = 0$$

(4.1)  

With such a choice for the $Q$ matrix and $\phi_1 = 3.55 \times 10^5$, the maximum response quantities are presented in the column, designated by "$P_e$", of Table 4-II. It should be emphasized that this is the simplest way of assigning the $Q$ matrix without involving any numerical computation. It can easily be shown that the control force $\mathbf{U}(t) = -\phi \mathbf{B} \mathbf{R}^{-1} \mathbf{Q} \mathbf{Z}(t)$ depends only on $\dot{x}_b$, i.e., the measurement of the relative velocity of the base isolation system. Consequently, only two velocity sensors are needed; one at the base isolation system and one at the foundation. No sensor is needed on the building.

It is observed from Table 4-II that the choice of the weighting matrix $Q$ using either $P_V$ or $\phi_2Q_1$ or $P_e$ are very satisfactory. Extensive numerical results indicate that the response of the building system can further be reduced with the increase of the active control force and that the aseismic hybrid control system is effective. For the particular hybrid control system considered, i.e., a rubber bearing isolation system connected to an actuator, the choice of the
Q matrix based on the energy concept is the simplest and preferable, because it requires only two velocity sensors. The reason that the energy approach works well for this particular situation is that for a base-isolated building, the deformation of the superstructure is small compared with the deformation of the base isolation system. As a result, the superstructure behaves like a rigid body.

Instead of using an actuator to protect the safety and integrity of the base isolation system, an active mass damper is connected to the base isolation system as shown in Fig. 3-1(b). The properties of the mass damper are as follows. The mass of the mass damper, m_d, is equal to 50% of the floor mass, i.e., m_d=0.5m_i and the natural frequency of the mass damper is the same as the first natural frequency of the base isolated building, i.e., 2.21 rad/sec. The damping ratio of the mass damper is 10%. With the active mass damper, the response vector is given by X(t)=[x_d, x_b, x_1, ..., x_8]' and the structural response depends on the weighting matrixes R and Q. For this example, the weighting matrix R consists of only one element, denoted by R_0. For illustrative purpose, \phi=\Delta t/3=0.0067 and R_0=0.0067 are used.

First, we consider the choice of \textbf{R}_v for the Q matrix, Eq. (2.22), in which I_1=Q and the diagonal elements of the diagonal matrix I_2 are I_{2ij}=3\times10^4 (j=1,2,...,9) and I_{2ij}=0 for j=10. Within 30 seconds of the earthquake episode, the maximum building response quantities and the maximum control force U_m are presented in the columns, designated by "\textbf{R}_v", of Table 4-II. The second case considered for the Q matrix is exactly the Riccati matrix \textbf{P}, Eq. (2.12). In this case, \phi_2=0.5 and \phi_3=2. The I_0 matrix used has the form given by Eq. (2.20) with I_1=Q and I_{2ij}=10^4 for j=2,3,...,10 and I_{2ij}=0 for j=1. The maximum response quantities are shown in the columns, designated as \textbf{P}, of Table 4-II. In the present case, the simple energy approach
does not perform well, because the purpose of the mass damper is to dissipate the energy of the building so that the kinetic energy of the mass damper should not be minimized.

It is observed from Table 4-II that (i) the aseismic hybrid control system using a lead-core rubber bearing isolation system and an active mass damper is very effective in protecting the building structure against strong earthquake, and (ii) the choice of the weighting matrix $Q$ using either the $P_v$ matrix or the Riccati matrix $P$ is very satisfactory.
SECTION 5

CONCLUSION

Within the framework of instantaneous optimal control, the assignment of the weighting matrix $Q$ is required to guarantee the stability of the controlled structure. The determination of the weighting matrix using the Lyapunov direct method is investigated for active and hybrid control systems. Various possible choices for the weighting matrix $Q$ and their performance are examined. For the particular linear structure equipped with an active bracing system considered herein, the choice of the $P_Y$ matrix seems to be quite attractive, since it involves the solution of linear algebraic equations.

For the building structure equipped with a rubber bearing isolation system connected to an actuator, the use of the energy concept for the $Q$ matrix is attractive and preferable. This is because it does not (i) require any sensor to be installed on the building, and (ii) involve any computational effort for the determination of the $Q$ matrix. For the building equipped with a rubber-bearing isolation system connected to an active mass damper, the choice of the $P_Y$ matrix, which involves the solution of linear algebraic equations, is quite satisfactory. The significant advantage of instantaneous optimal control in conjunction with the Lyapunov approach has been demonstrated for applications to hybrid control systems, which involve active control of nonlinear or hysteretic structural systems.
REFERENCES


APPENDIX I

ALTERNATIVE DERIVATION OF INSTANTANEOUS
OPTIMAL CLOSED-LOOP CONTROL.

Given the time dependent performance index \( J(t) \) in Eq. (2.6) and the matrix equation of motion in Eq. (2.3), the optimal feedback control vector is assumed to be a linear function of \( Z(t) \),

\[
U(t) = GZ(t)
\]

(I-1)

in which \( G \) is a \((r \times 2n)\) gain matrix to be determined. Thus, the gain matrix \( G \) should guarantee the stability of the controlled structure. Substitution of Eq. (I-1) into Eq. (2.3) leads to the equation of motion \( \dot{Z}(t) = (A + B \cdot G)Z(t) \) in which the excitation \( \dot{X}_0(t) \) has been dropped, since it does not affect the gain matrix \( G \). The transition solution of the (stable) controlled structure can be expressed as

\[
Z(t) = \exp[(A + B \cdot G)\Delta t] Z(t-\Delta t)
\]

(I-2)

in which \( Z(t-\Delta t) \) is the response state vector at the time \( t-\Delta t \), and \( \Delta t \) is a small time interval.

Substituting Eqs. (I-1) and (I-2) into Eq. (2.6), one obtains

\[
J(t) = \dot{Z}'(t-\Delta t) S Z(t-\Delta t)
\]

(I-3)

in which

\[
S = \exp[A' \cdot \Delta t + G' \cdot B' \cdot \Delta t] (Q + G' \cdot R \cdot G) \exp[A \cdot \Delta t + B \cdot G \cdot \Delta t]
\]

(I-4)

The necessary conditions for minimizing \( J(t) \), Eq. (I-3), are \( \partial J(t)/\partial g_{ij} = 0 \) for \( i=1,2,\ldots,r \) and \( j=1,2,\ldots,2n \), where \( g_{ij} \) is the \( i\)-\( j \) element of the gain matrix \( G \). It follows from Eq. (I-3) that these necessary conditions are equivalent to the following conditions

\[
\frac{\partial S}{\partial g_{ij}} = 0 \text{ for } i=1,2,\ldots,r \text{ ; } j=1,2,\ldots,2n
\]

(I-5)

Substituting Eq. (I-4) into Eq. (I-5) and making appropriate arrangements, one obtains

\[
E_{ij}'[B'(Q + G' \cdot R \cdot G) \Delta t + R \cdot G] + [B'(Q + G' \cdot R \cdot G) \Delta t + R \cdot G]'E_{ij} = 0
\]

(I-6)

in which \( E_{ij} = \partial G/\partial g_{ij} \). Note that the first term of Eq. (I-6) is equal to the transpose of the second term. Since \( E_{ij} \neq 0 \) for \( i=1,2,\ldots,r \) and \( j=1,2,\ldots,2n \), one obtains

\[
B'(Q + G' \cdot R \cdot G) \Delta t + R \cdot G = 0
\]

(I-6)
For small $\Delta t$, the gain matrix is obtained as

$$G = -\Delta t R^{-1} B' Q$$  \hspace{1cm} (I-7)

Since the gain matrix $G$ should guarantee the stability of the controlled structure, it follows from Eq. (I-7) that the weighting matrix $Q$ should also guarantee the stability of the controlled structure. Likewise, the gain matrix $G$ should be finite.
APPENDIX II
EQUIVALENT LINEARIZATION AT Z=0

With the assumption of linear viscous damping for the structural system, the damping vector \( \mathbi{F}_D(\dot{\mathbi{X}}) \) in Eq. (3.2) can be expressed as \( \mathbi{C}_1\dot{\mathbi{X}} \), i.e., \( \mathbi{F}_D(\dot{\mathbi{X}}) = \mathbi{C}_1\dot{\mathbi{X}} \), where the (nxn) damping matrix \( \mathbi{C}_1 \) has been described in the text. It follows from Eqs. (3.12) and (3.13) that the stiffness vector \( \mathbi{F}_k(\mathbi{X}) \) in Eq. (3.2) can be expressed as

\[
\mathbi{F}_k(\mathbi{X}) = \mathbi{K}_e \dot{\mathbi{X}}(t) + \mathbi{K}_f \mathbi{V}(t) \tag{II-1}
\]

in which \( \mathbi{V}(t) = [v_1, v_2, \ldots, v_n]' \) = an n vector denoting the hysteretic variable of each story unit, and \( \mathbi{K}_e \) and \( \mathbi{K}_f \) are (nxn) band-limited elastic stiffness matrix and hysteretic stiffness matrix, respectively. All elements of \( \mathbi{K}_e \) and \( \mathbi{K}_f \) are zero except \( K_e(i,i) = \alpha_i k_i \), \( K_f(i,i) = (1-\alpha_i)k_i D_yi \) for \( i=1,2,\ldots,n \) and \( K_e(i,i+1) = -\alpha_i k_{i+1} \), \( K_f(i,i+1) = -(1-\alpha_i)k_{i+1} D_yi+1 \) for \( i=1,2,\ldots,n-1 \).

Thus, the nonlinear vector \( \mathbi{g}[Z] \) given by Eq. (3.2) can be expressed as

\[
\mathbi{g}[Z] = \begin{bmatrix}
\dot{\mathbi{X}} \\
-\mathbi{M}_1^{-1}[\mathbi{C}_1 \ddot{\mathbi{X}} + \mathbi{K}_e \dot{\mathbi{X}} + \mathbi{K}_f \mathbi{V}]
\end{bmatrix} = \begin{bmatrix}
g_1 \\
g_2
\end{bmatrix} \tag{II-2}
\]

where \( g_1 = \dot{\mathbi{X}} \) and \( g_2 = -\mathbi{M}_1^{-1}[\mathbi{C}_1 \ddot{\mathbi{X}} + \mathbi{K}_e \dot{\mathbi{X}} + \mathbi{K}_f \mathbi{V}] \). It follows from Eqs. (3.12) and (3.13) that \( \mathbi{V}(t) \) is equal to zero for \( Z=0 \). Consequently, the derivative matrix \( \Delta(Z) \), Eq. (3.7), evaluated at \( Z=0 \), Eq. (3.11), is given by

\[
\Delta = \Delta(Z)|_{Z=0} = \begin{bmatrix}
\frac{\partial g_1}{\partial \mathbi{X}} & \frac{\partial g_1}{\partial \dot{\mathbi{X}}} \\
\frac{\partial g_2}{\partial \mathbi{X}} & \frac{\partial g_2}{\partial \dot{\mathbi{X}}}
\end{bmatrix}_{Z=0, K=0} \tag{II-3}
\]

in which \( g_1 \) and \( g_2 \) are given by Eq. (II-2). Substituting the expressions of \( g_1 \) and \( g_2 \) into Eq. (II-3) and using Eqs. (3.12) and (3.13), one obtains the resulting (2nx2n) matrix \( \Delta \) which is given by Eq. (3.14).